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HARDY INEQUALITIES, CONTROL AND NUMERICS FOR SINGULAR PDEs

Memoria para optar al título de Doctor en Ciencias Matemáticas

presentada por

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Contents

Resumen

En esta Tesis se estudian desigualdades óptimas de Hardy para operadores de Schrödinger con singularidades cuadráticas y se describe su contribución al estudio de las propiedades cualitativas como la regularidad, el control y los aspectos numéricos de las ecuaciones en derivadas parciales singulares.

En las desigualdades de Hardy en consideración desarrollamos resultados óptimos para varios tipos de potenciales con singularidades localizadas en un número finito o infinito de polos singulares. Se analizan singularidades situadas en el interior o en la frontera de dominios tanto regulares como no-regulares (por ejemplo, dominios cónicos). En términos generales, desigualdades mejoradas se producen cuando se trata de singularidades localizadas en la frontera en vez del interior. Nuestras demostraciones se basan en transformaciones convenientes o en cambios de variables adaptados a la geometría del dominio y a la localización de los polos. También aplicamos la descomposición en armónicos esféricos para mostrar resultados óptimos en dominios regulares. Para dominios cónicos, mostramos límites inferiores de las correspondientes constantes óptimas que mejoran algunos de los resultados existentes en la literatura. Las pruebas de los últimos resultados requieren propiedades finas de las funciones de Bessel.

Para el operador de Schrödinger $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda > 0$, en primer lugar verificamos la validez de una identidad de tipo Pohozaev cuando la singularidad se coloca en la frontera de un dominio regular. A continuación, aplicamos nuestra identidad de Pohozaev para ecuaciones elípticas semi-lineales, y también para desarollar el método de multiplicadores que corresponde a la controlabilidad exacta para las ecuaciones de ondas y Schrödinger singulares.

Para la ecuación del calor asociada al operador A_λ, cuya singularidad se encuentra en la frontera de un dominio regular, analizamos el problema de controlabilidad con un control interno. Este problema también ha sido estudiado en el pasado en el caso de una singularidad interna. En nuestro análisis se aplican estimaciones de Carleman con una elección de los pesos distinta de los que se utilizaron en los trabajos previos en el caso de la singularidad interna.

Para ambas ecuaciones anteriores (de ondas y de calor) involucrando el operador A_λ, las propiedades de controlabilidad están muy relacionadas a la constante óptima de Hardy con una singularidad en la frontera, $\lambda(N) = N^2/4$, donde $N \geq 1$ denota la dimensión del dominio. En términos generales, nuestro trabajo complementa y extiende al nuevo rango de parámetros $\lambda \leq N^2/4$, resultados previos válidos para una singularidad interna en el rango

$\lambda \leq \lambda_\star := (N-2)^2/4$. En este último caso, es bien conocido que la constante óptima de Hardy es $(N-2)^2/4$.

Para el problema elíptico correspondiente a A_λ con condiciones de frontera de Dirichlet, en un dominio $\Omega \subset \mathbb{R}^N$ que contiene el origen, ya sea en el interior o en el borde, se muestra la falta de regularidad elíptica estándar de las soluciones. Para el problema unidimensional con datos en $L^2(\Omega)$, la solución pertenece a $H^{1+s}(\Omega)$ para cualquier valor $s < s(\lambda) < 1$, donde $s(\lambda)$ es una constante que converge a cero cuando λ tiende a la constante crítica de Hardy. Se obtienen resultados similares en el caso multidimensional en la clase de soluciones radiales mediante una reducción al problema unidimensional. Para probar esto, aplicamos transformaciones logarítmicas y el método de la variación de las constantes para determinar fórmulas explícitas para las soluciones. De manera más precisa probamos resultados de regularidad en espacios de Sobolev fraccionarios. Por otra parte, para soluciones no-radiales mostramos un comportamiento asintótico cerca del origen de sus componentes radiales. Esto se hace a través de un análisis espectral y propiedades finas de las funciones de Bessel.

En el contexto numérico, analizamos el funcionamiento de los métodos de elementos finitos (FEM) usando elementes lineales a trozos P1, para el problema elíptico asociado a A_λ. Al analizar las tasas de convergencia para el FEM estándar con mallados uniformes, se observa que, debido a la presencia de la singularidad, la regularidad elíptica estándar falla y el FEM clásico ofrece tasas de convergencia más débiles que para el Laplaciano clásico (que corresponde a $\lambda = 0$). En términos generales, cuanto menos regular es la solución, menos velocidad de convergencia se tiene.

Para compensar la falta de regularidad del problema, abordamos la cuestión de construir aproximaciones FEM usando mallados heterogéneos adaptados a la singularidad. El objetivo consiste en obtener un error prescrito en la norma de la energía utilizando un número mínimo de iteraciones. Para este, utilizamos el denominado Adaptive FEM (AFEM), que es una extensión de la clásica FEM y probabos que este método mejora las tasas de convergencia mediante el uso de mallas no uniformes que están adaptados al potencial singular. Para el problema en una dimensión, se utilizan estimaciones a priori para mostrar las tasas óptimas de convergencia. Para el problema multi-dimensional, utilizamos para construir un algoritmo adaptativo y mostramos algunos experimentos numéricos basados en dicho algoritmo. Identificamos algunas mallas patológicas entorno a la singularidad.

Abstract

In this Thesis we study optimal constants in Hardy inequalities for Schrödinger operators with quadratic singular potentials, and we describe their contribution to the study of qualitative properties as regularity, controllability and numerical aspects of singular PDEs.

In the Hardy inequalities under consideration we develop optimal results for several types of potentials with quadratic singularities located at one or finite or even infinite number of singular poles. We analyze singularities located in the interior or on the boundary of both smooth and non-smooth domains (e.g. conical domains). Roughly speaking, better inequalities occur when we deal with singularities located on the boundary rather than in interior. Our proofs are based on convenient transformations or change of variables methods adapted to the geometry of the domain under consideration and the localization of the singular poles. We also apply spherical harmonic decompositions to show optimal results in smooth domains. For conical domains, we show lower bounds for the corresponding optimal constants which improve some of the results existing in the literature. The proofs of the latter results require fine properties of Bessel functions.

For the Schrödinger operator $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda > 0$, we first verify the validity of a Pohozaev-type identity in the situation where the origin is located on the boundary of a smooth domain. Then, we apply our Pohozaev identity to semi-linear elliptic equations but we also employ it to derive the method of multipliers which yields the exact boundary controllability of both the wave and Schrödinger equations corresponding to the singular operator A_λ.

For the heat equation with potential $-\lambda/|x|^2$, $\lambda > 0$ whose singularity is located on the boundary of a smooth domain, we analyze the controllability problem with a distributed control. This problem has also been studied in the past in the case of interior singularity. Our analysis applies Carleman estimates for different choice of weights than those used in the previous works in the case of interior singularity.

For both wave and heat equations corresponding to A_λ above, the control properties are very much related to the optimal constant in the Hardy inequality with boundary singularity, namely $\lambda(N) = N^2/4$, where $N \geq 1$ denotes the dimension of the domain. Roughly speaking, our work complements and extends to the new range of parameters $\lambda \leq N^2/4$, previous control results obtained in the case of interior singularity in the range $\lambda \leq \lambda_\star := (N-2)^2/4$, where $(N-2)^2/4$ is the optimal Hardy constant with interior singularity.

For the elliptic problem corresponding to A_λ with Dirichlet boundary conditions, in a domain $\Omega \subset \mathbb{R}^N$ for any $N \geq 1$, which contains the origin either in the interior or on the boundary, we show the lack of standard elliptic regularity for the solutions. For the one dimensional problem and with data in $L^2(\Omega)$, the solution belongs to $H^{1+s}(\Omega)$ for any value $s < s(\lambda) < 1$, where $s(\lambda)$ is a constant which converges to zero as λ tends to the critical Hardy constant. We obtain similar results in the multi dimensional case in the class of radial solutions by using a one dimensional reduction to the problem. To prove this, we apply logarithmic transformations and the variation of constants formula to determine explicit formulas for the solutions. Estimates in fractional Sobolev spaces yield the regularity results. Moreover, for non-radial solutions we show asymptotic behavior near the origin of their radial component. This is done by means of spectral analysis and fine properties of Bessel functions.

In the numerical context we analyze the performance of finite element methods (FEM) based on P1 piecewise linear elements, when solving the Dirichlet problem corresponding to the singular elliptic operator A_λ. When analyzing the rates of convergence for the standard FEM with uniform-meshes, we observe that, due to the presence of the singularity, standard elliptic regularity fails and the classical FEM provides weaker convergence rates than for the Laplacian (which corresponds to $\lambda = 0$). Loosely speaking, the less regular the continuous solution is, the less the error converges to zero.

To compensate the lack of regularity of the problem, we address the question of constructing FEM approximations with non-heterogeneous meshes adapted to the singularity, in order to obtain a prescribed error in the energy norm using minimum possible number of iterations. To this end, we use the so-called Adaptive FEM (AFEM) which is an extension of the classical FEM. Hence, we improve the convergence rates by using non-uniform meshes which are adapted to the singularity of the potential. For the one dimensional problem, we use a priori estimates proportional to the asymptotic decay at the origin, to show optimal convergence rates. For the multi-dimensional problem, in the case of boundary singularity, we show some numerical experiments based on an posteriori analysis in which we remark some pathological mesh refinements in the neighborhood of the singularity.

Chapter 1

Introducción

Motivación y comentarios bibliográficos

En la teoría de las ecuaciones en derivadas parciales singulares, la desigualdad de Hardy ha jugado un papel crucial en el análisis y descripción de las propiedades cualitativas de los operadores de Schrödinger con potenciales de cuadrado inverso.

En 1925, G. H. Hardy [79] probó que, en el caso de una sola dimensión, se cumple

$$\forall u \in H_0^1(0,\infty), \qquad \int_0^\infty |u'(r)|^2 dr \geq \frac{1}{4}\int_0^\infty \frac{u^2(r)}{r^2} dr. \tag{1.0.1}$$

Pocos años más tarde, la desigualdad (1.0.1) fue mencionada también en [Leray [85], pág. 204], donde el autor estudió la existencia de soluciones periódicas para fluidos viscosos no-acotados.

Más tarde, (1.0.1) fue generalizada al caso multidimensional en Hardy-Littlewood-Pólya [78]. La llamada clásica desigualdad de Hardy afirma que, para cualquier subconjunto abierto $\Omega \subset \mathbb{R}^N$, $N \geq 3$, que contiene el origen, y para cualquier $u \in H_0^1(\Omega)$, se tiene que $u/|x| \in L^2(\Omega)$ y la siguiente estimación:

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx. \tag{1.0.2}$$

La constante $(N-2)^2/4$ en (1.0.2) es óptima y no se alcanza en $H_0^1(\Omega)$. En otras palabras, la inclusión continua $H_0^1(\Omega) \subset L^2(\Omega, |x|^{-2}dx)$ no es compacta.

El estudio de la desigualdad de Hardy (1.0.2) está motivado por sus aplicaciones a diversos campos. En particular, es conocido por los físicos cuánticos que la desigualdad (1.0.2) tiene estrecha relación con el Principio de incertidumbre de Heisenberg (véase, por ejemplo Fefferman [61]).

Además, se puede aplicar tanto para obtener cotas inferiores para el espectro de operadores (generalizados) de Schrödinger de la forma $-\Delta - V(x)$ (para los potenciales $V(x) > 0$, con singularidades aisladas, $V \in L^1_{loc}(\Omega)$) con condiciones de contorno de Dirichlet, cómo para analizar el comportamiento del límite de los núcleos de calor (ver Davis [45]).

La desigualdad de Hardy juega también un papel importante en la Mecánica Cuántica, a la hora de estudiar la ecuación de Schrödinger no-relativista para una sola partícula que se mueve en un campo eléctrico

$$Eu = -\frac{\hbar^2}{2\mu}\Delta u - \frac{e^2}{4\pi\varepsilon_0 r}u. \tag{1.0.3}$$

La ecuación (1.0.3) aparece en la descripción del átomo de hidrógeno (véase, por ejemplo [6]), donde E es una constante de proporcionalidad, $\hbar$ es la constante de Planck, e es la carga del electrón, $r = |x|$ es la posición, ε_0 es la constante eléctrica (permitividad del espacio libre) y $\mu = m_e m_p/(m_e + m_p)$ es la masa reducida del núcleo de hidrógeno (un protón) de la masa m_p y el electrón de masa m_e.

El potential de Hardy $V = |x|^{-2}$ en (1.0.2) tiene una homogeneidad igual a -2, que es crítica desde el punto de vista matemático y físico. Debido a su criticidad, no pueden ser considerados como perturbaciones de orden inferior del Laplaciano.

Además, los potenciales con singularidades críticas también se aplican en los modelos cosmológicos cuánticos tal y como se destaca en la ecuación de Wheeler-de-Witt (véase, por ejemplo [14]), o en la linealización de ecuaciones en derivadas parciales no lineales que describen las características de las soluciones para modelos que se plantean en la teoría de combustión (por ejemplo, Gelfand [71]).

Vamos a motivar estos hechos considerando la EDP elíptica semi-lineal

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.0.4}$$

donde $\Omega \subset \mathbb{R}^N$ con $N \geq 3$ es un dominio abierto acotado. La función f no-lineal se supone que es una función continua, positiva, creciente y convexa de modo que $f(0) > 0$ y

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty. \tag{1.0.5}$$

La ecuación (1.0.4) aparece cuando se estudia la descripción de una bola de gas isotérmica en el equilibrio gravitatorio propuesto por Lord Kelvin [41].

Es bien sabido por Brezis-Cazenave [22] y Brezis-Nirenberg, [25] que existe un número positivo finito $\lambda^\star = \lambda^\star(\Omega)$ llamado *valor extremo* tal que, para cualquier $0 \leq \lambda < \lambda^\star$, el problema (1.0.4) tiene una solución clásica $\underline{u}_\lambda \in C^2(\overline{\Omega})$, que es *mínima* entre todas las posibles soluciones. De lo contrario, si $\lambda > \lambda^\star$, el problema de Dirichlet (1.0.4) no tiene soluciones débiles (ni siquiera en $L^1(\Omega)$).

Después, Brezis y Vázquez demostraron en [26] que, bajo la condición de (1.0.5) en f, el siguiente límite existe en casi todos puntos:

$$u^\star(x) := \lim_{\lambda \to \lambda^\star} \underline{u}_\lambda(x) \tag{1.0.6}$$

y $u^\star \in L^1(\Omega)$ es una solución débil del problema (1.0.4), la llamada *solución extremal* correspondiente a $\lambda^\star$. Dado que las propiedades de las soluciones clásicas extremales han sido bien estudiadas, los autores en [26] estuvieron interesados en la caracterización de las soluciones extremales no-acotadas $u^\star$ (en la clase de energía $H_0^1(\Omega)$) y en el valor extremo $\lambda_\star$.

En Brezis-Vázquez [26] también se demostró que $u^\star \in H_0^1(\Omega)$ es una solución extremal no-acotada para (1.0.4) correspondiente a $\lambda = \lambda^\star$, si y sólo si el primer autovalor del operador lineal $-\Delta - \lambda^\star f'(u^\star)$ es no-negativo, es decir,

$$\int_\Omega |\nabla u|^2 dx \geq \lambda^\star \int_\Omega f'(u^\star) u^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{1.0.7}$$

En particular, si $f(u) = e^u$, esto corresponde al célebre problema de Gelfand [72]. Si por otra parte, $\Omega = B_1(0)$ es la bola unidad en $\mathbb{R}^N$, entonces, de manera explícita se puede deducir que $(\lambda^\star, u^\star) = (2(N-2), -2\log(|x|))$ para cualquier $N \geq 10$, puesto que la desigualdad

$$\int_\Omega |\nabla u|^2 dx \geq 2(N-2) \int_\Omega \frac{u^2}{|x|^2} dx \tag{1.0.8}$$

es cierta, debido a la desigualdad de Hardy (1.0.2).

Si $N \leq 9$ se muestra en [26] que no hay soluciones para el problema extremal Gelfand [72].

Otro ejemplo típico de f en (1.0.4) es $f(u) = (1+u)^p$, $p > 1$ (véase, por ejemplo [75], [98]).

Para una visión general sobre la literatura existente sobre la desigualdad de Hardy nos referimos a los artículos de revisión de Davies [46] y Brezis-Marcus [23].

El lector interesado en las extensiones de la clásica desigualdad de Hardy (1.0.2) se puede referir, en particular, a los siguientes artículos y referencias: [26], [71], [4], [27], [10], [65], [105], [111], [104], [84], [48], [51], [50], [66].

Recientemente, versiones mejoradas de (1.0.1)-(1.0.2) han sido establecidas en dominios acotados abiertos que contienen el origen (ver [26], [3], [60]). La generalización de la desigualdad de Hardy en el caso del $p-$Laplaciano se ha obtenido recientemente en [58]. Mencionamos también los trabajos [19], [64] y las referencias en el mismo para problemas con potenciales multipolares. Ha habido también una amplia investigación sobre problemas con potenciales involucrando la distancia a la frontera (por ejemplo, [23], [24]).

El contenido de la Tesis

En el Capítulo 2 de la tesis se estudian desigualdades de Hardy para operadores de Schrödinger con potenciales cuyas singularidades se encuentran ya en el interior o en la frontera de un dominio (acotado o no acotado, regular o no suave) abierto. Se analizan los casos de potenciales con uno, varios o infinito número de polos.

Éstos resultados sobre las desigualdades de Hardy se utilizarán en los siguientes capítulos de la tesis en varias aplicaciones como problemas de control de las ecuaciones de ondas y del calor con potenciales singulares, o en el diseño de métodos numéricos eficientes para problemas elípticos con singularidad cuadrática.

Por lo tanto, en los Capítulos 3 y 4 se analizarán las propiedades de control de las ecuaciones de evolución (tanto hiperbólicas y cómo parabólicas), con un potencial singular localizado en el borde de un dominio regular y más precisamente, analizamos cómo la singularidad afecta en la existencia y unicidad de los problemas, en los aspectos de regularidad, etc...

En el contexto del problema de controlabilidad para la ecuación de ondas con un potencial singular ubicado en la frontera, que estudiamos en el Capítulo 3, es necesario demostrar una desigualdad de observabilidad para la solución del sistema adjunto en la cual la energía total se recupera a través de la derivada normal en la frontera, en una norma L^2 ponderada y degenerada en el origen. Para ello, primero hay que justificar alguna propiedad de regularidad oculta para la derivada normal de la solución del sistema adjunto para datos iniciales en el espacio de energía.

Esto no es una consecuencia directa del hecho de que el problema está bien propuesto en el espacio de energía, sino que se debe a las propiedades intrínsecas de la ecuación de ondas y a la presencia del potencial singular en la frontera.

En el Capítulo 5 estudiamos la regularidad del operador de Schrödinger con potenciales singulares. En primer lugar, analizamos el caso de una sola dimensión, cuando la singularidad se encuentra en la frontera y después el caso multi-dimensional con una singularidad interna.

Más tarde, en el Capítulo 6 nos encontramos con el desarrollo y el análisis de métodos de aproximaciones numéricas adaptativos de dichas ecuaciones en derivadas parciales singulares, en el contexto de los Métodos de Elementos Finitos (FEM).

Como las singularidades críticas afectan a la regularidad de las soluciones de las ecuaciones en derivadas parciales, la teoría elíptica clásica no es aplicable y, esto, como veremos, implica que no se puede transladar ni en el contexto continuo ni para los métodos de aproximaciones numéricas.

Los principales resultados de esta tesis se pueden clasificar de la siguiente manera:

1. El Capítulo 2: Desigualdades de Hardy

   1). Potencial singular cuadrático ubicado en el frontera: demostramos cotas inferiores para las constantes óptimas de Hardy en dominios cónicos, que mejoran algunos resultados conocidos en la literatura, y se muestran desigualdades óptimas en dominios regulares que proporcionan mejores resultados que la desigualdad de Hardy clásica (1.0.2).

   2). Potenciales multipolares: determinamos las constantes óptimas para desigualdades con potenciales que tienen un número finito de singularidades cuadráticas, independientemente de la ubicación de los polos. Los resultados responden a la cuestión de la optimalidad de algunas desigualdades discutidas en Bosi-Dolbeault-Esteban [19].

   3). Potenciales con un número infinito de singularidades: determinamos límites inferiores no triviales para desigualdades en dominios cilíndricos con un número infinito de singularidades localizadas periódicamente en el eje del cilindro. Por lo que sabemos, estos resultados son completamente nuevos.

2. El Capítulo 3: Controlabilidad para las ecuaciones de ondas y Schrödinger con potenciales singulares en la frontera.

   Probamos resultados de controlabilidad para las ecuaciones de ondas y Schrödinger con un potencial úbicado en la frontera de un dominio regular. Ampliamos el análisis en Vaconstenoble-Zuazua [109] en el contexto de las singularidades interiores, al caso de una singularidad en el borde. Nuestros resultados dan lugar a una desigualdad de observabilidad más fina, con el fin de compensar la falta de regularidad en la frontera.

3. El Capítulo 4: Controlabilidad nula para la ecuación del calor con un potencial singular en la frontera

Probamos resultados de controlabilidad para la ecuación del calor con potencial singular en el borde, actuando con controles internos. Utilizando las estimaciones de Carleman, confirmamos y ampliamos resultados similares de Ervedoza [54] de una singularidad interna, al caso de una singularidad en el borde. Nuestros resultados se basan en una modificación adecuada de los pesos de las estimaciones de Carleman [54].

4. El Capítulo 5: Operadores de Schrödinger con condiciones de contorno Dirichlet y potenciales singulares internos

   Probamos la regularidad de soluciones radiales y determinamos el comportamiento asintótico de la parte radial de las soluciones no radiales del problema en estudio. Por lo que sabemos, nuestro análisis es nuevo.

5. El Capítulo 6: Aproximaciones numéricas para operadores de Schrödinger con potenciales de cuadrado inverso.

   Probamos aproximaciones de elementos finitos para el problema unidimensional a través de un análisis a priori. Por otra parte, en dos dimensiones, presentamos experimentos numéricos desarrollando códigos FEM adaptativos usando análisis a posteriori cuando el potential actúa en la frontera. Los resultados numéricos confirman los aspectos teóricos desarrollados en esta tesis.

En la segunda parte se describen brevemente con más detalle nuestros principales resultados y las técnicas que hemos desarrollado.

Capítulo 2: Desigualdades de Hardy

En el Capítulo 2 de esta tesis analizamos la positividad (en formas cuadráticas L^2) de operadores de Schrödinger $-\Delta - V(x)$, $V(x) > 0$, con potenciales de cuadrado inverso de la forma

$$A_\lambda := -\Delta - \frac{\lambda}{|x|^2}, \tag{1.0.9}$$

$$B_\lambda := -\Delta - \lambda \sum_{1\leq i<j\leq n} \frac{|x_i - x_j|^2}{|x-x_i|^2|x-x_j|^2}, \tag{1.0.10}$$

y

$$C_\lambda := -\Delta - \sum_{i\in I} \frac{a_i}{|x-x_i|^2}, \tag{1.0.11}$$

donde $\lambda > 0$, $n \in \mathbb{N}$ y $n \geq 1$ son números enteros y I es un conjunto numerable de índices tales que $a_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$ con $i \in I$. Estos potenciales pueden tener uno, un número finito o un número infinito de polos singulares.

Estamos especialmente interesados en la desigualdad de Hardy óptima

$$\int_\Omega |\nabla u|^2 dx \geq \mu(\Omega) \int_\Omega V(x) u^2 dx, \quad \forall u \in H_0^1(\Omega), \tag{1.0.12}$$

donde Ω es un abierto de $\mathbb{R}^n$, $n \geq 1$, y la constante óptima $\mu(\Omega)$ está definida como

$$\mu(\Omega) := \inf \Big\{ \int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega V(x) u^2 dx, \; u \in H_0^1(\Omega) \Big\}.$$

Por supuesto, el problema vuelve ser interesante cuando los polos de los potenciales V en (1.0.9)-(1.0.11) pertenecen a $\overline{\Omega}$. De lo contrario es una perturbación regular del Laplaciano.

En el Capítulo 2 abordamos la cuestión de determinar el valor $\mu(\Omega)$ (o, alternativamente, mostramos límites superiores e inferiores), en términos de la naturaleza de los potenciales V en cada una de las situaciones (1.0.9)-(1.0.11).

A continuación se describen brevemente los resultados principales del Capítulo 2.

1. **Singularidades de frontera.**

   En la Sección 2.3 hablamos de la positividad de los operadores singulares de la forma

   $$A_\lambda = -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda > 0,$$

   en los que la singularidad $x = 0$ se coloca en la frontera de $\partial\Omega$ de un dominio abierto $\Omega \subset \mathbb{R}^N$ y $N \geq 2$. Analizamos dos tipos de configuraciones geométricas de Ω dependiendo de la regularidad en el origen.

   - *Dominios regulares (o dominios Lipschitz de clase C^2 cerca de $x = 0$).* En términos generales, mostramos que el valor de $\mu(\Omega)$ salta de $(N-2)^2/4$, la constante clásica en el caso de la desigualdad de Hardy en dominios con singularidad interna, a la nueva constante de Hardy $N^2/4$. Esto se logró por primera vez para $\Omega = \mathbb{R}^N_+$, donde $\mathbb{R}^N_+$ es la mitad del espacio de $\mathbb{R}^N$ en el que la condición $x_N > 0$ se cumple. Más precisamente, para cualquier $N \geq 2$, Filippas, Tertikas y Tidblom demostraron en [67] la nueva desigualdad óptima de Hardy

     $$\forall u \in H_0^1(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{u^2}{|x|^2} dx. \tag{1.0.13}$$

     En la Sección 2.3.1 probamos la desigualdad (1.0.13) en cualquier dominio acotado regular Ω que contiene el origen en la frontera. Estos resultados están estrechamente relacionados con la convexidad global de Ω, pero son independientes de la geometría de Ω en el origen $x = 0$ (tal como se destaca en la Figura 1.1).

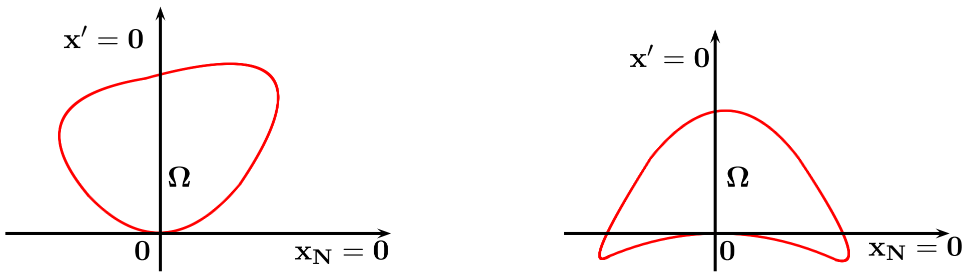

Figure 1.1: Geometría convexa (a la izquierda), falta de convexidad (a la derecha).

Más precisamente, independiente de la geometría de Ω, existe una constante $r_0 > 0$ que depende de Ω tal que

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}, \tag{1.0.14}$$

donde $B_{r_0}(0) \subset \mathbb{R}^N$ es la bola de radio r_0 centrada en en el origen. Como consecuencia de ello tenemos que $\mu(\Omega) \leq N^2/4$. En particular, si Ω es convexo se cumple que $\mu(\Omega) = N^2/4$. De lo contrario, señalamos que, para algunas geometrías particulares de Ω, es necesario añadir un término de orden inferior (en la norma L^2) para mantener la desigualdad:

$$\forall u \in H_0^1(\Omega), \quad C\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx, \tag{1.0.15}$$

donde $C > 0$ es una constante positiva dependiendo de Ω. Existen dominios no-convexos para los cuales el término de orden más bajo en (1.0.15) puede ser eliminado pero hay contraejemplos también (véase la Proposición 2.3.4). Además, en los dominios acotados, podemos mejorar estos resultados abordando la cuestión de la búsqueda de cotas inferiores no triviales para la diferencia

$$I[u] = \int_\Omega |\nabla u|^2 dx - \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx,$$

que son las llamadas desigualdades del tipo Hardy-Poincaré.

Parte de estos resultados fueron enunciados por primera vez en Cazacu-Zuazua [39] y Cazacu [38]. Resultados similares se obtuvieron de forma independiente por Fall y Musina [59], [57]. En la Sección 2.3.1 presentamos nuestras aportaciones complementarias a las del [59], [57], y también damos algunas pruebas alternativas o simplificadas.

- *Dominios no regulares (dominios cónicos con el vértice situado en $x = 0$).* Otra situación interesante es aquella en la cuál la frontera del dominio desarrolla esquinas en la singularidad. El ejemplo más relevante es el de un dominio cónico con el vértice en el origen $x = 0$ (Ver Figura 1.2).

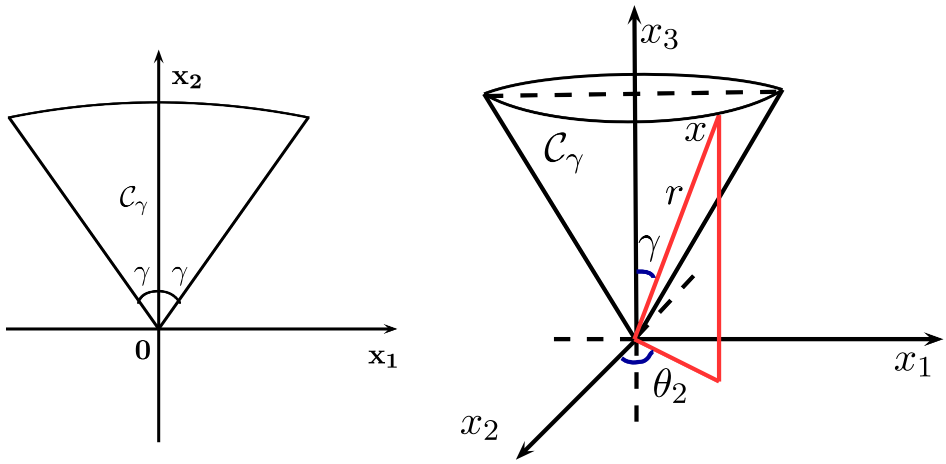

Figure 1.2: El sector bidimensional con la apertura γ (a la izquierda). El cono tridimensional con la apertura γ (a la derecha).

Consideramos dominios cónicos $\Omega = \mathcal{C}_\gamma \subset \mathbb{R}^N$, $N \geq 2$, con el vértice en $x = 0$ y la amplitud de $\gamma \in (0, \pi)$ (véase la Figura 1.2).

Es bien sabido que (véase, por ejemplo Tintarev [93])

$$\mu(\mathcal{C}_\gamma) = \frac{(N-2)^2}{4} + \lambda_1(\gamma), \tag{1.0.16}$$

donde $\lambda_1(\gamma)$ es el primer autovalor del operador Laplace-Beltrami en la superficie $\mathcal{C}_\gamma \cap S^{N-1}$, donde S^{N-1} denota la esfera unidad en $\mathbb{R}^N$ con la medida de Hausdorff $(N-1)$.

La cuestión de estudiar el valor exacto de $\mu(\Omega)$ en los dominios cónicos se ha abordado en una serie de casos (véase, por ejemplo [15], [30]). Si $N = 2$ se tiene $\mu(\mathcal{C}_\gamma) = \pi^2/(2\gamma)^2$. Si $N = 4$ entonces $\mu(\mathcal{C}_\gamma) = \pi^2/\gamma^2$. Si $N = 3$ o $N \geq 5$, el valor de $\lambda_1(\gamma)$ en (1.0.16) no se ha determinado de forma explícita. A pesar de ello, límites inferiores y superiores han sido calculados.

Por lo que conocemos, las mejores cotas inferiores se hallaron en [30]. En particular, si $N = 3$ y $\gamma \in (0, \pi/2)$ entonces (cf. [94], [30])

$$\lambda_1(\gamma) \geq \frac{j_0^2}{\gamma^2} - \left(\frac{1}{2} - \frac{1}{\pi^2}\right) \tag{1.0.17}$$

donde j_0 denota el primer cero de la función de Bessel J_0. Además, si $N \geq 5$ y $\gamma \in (0, \pi/2)$ entonces (cf. [30])

$$\lambda_1(\gamma) \geq \frac{j_{(N-3)/2}^2}{\gamma^2} - \frac{(N-2)(N-1)}{6} \tag{1.0.18}$$

donde $j_{(N-3)/2}$ es el primer cero positivo de la función de Bessel $J_{(N-3)/2}$.

En la Sección 2.3.2, para cualquier dimensión $N \geq 2$, demostramos que

$$\lambda_1(\gamma) \geq (N-1)\frac{\pi^2}{4\gamma^2}. \tag{1.0.19}$$

En vista de (1.0.19), nuestro resultado mejora los límites inferiores (1.0.17)-(1.0.18) en [30]. Además, en la Sección 2.3.2 damos una prueba alternativa para el valor asintótico de $\lambda_1(\gamma)$ en [30], cuando γ tiende a cero, que es

$$\lim_{\gamma \to 0} \lambda_1(\gamma)\gamma^2 = j_{(N-3)/2}^2, \quad \forall N \geq 3. \tag{1.0.20}$$

En particular, en cualquier dimensión $N \geq 3$, probamos el límite inferior

$$\lambda_1(\gamma) \geq \left(\frac{\sin\gamma}{\gamma}\right)^{N-2} \frac{j_{(N-3)/2}^2}{\gamma^2}, \quad \forall \gamma \in (0, \pi), \tag{1.0.21}$$

que, para γ pequeño, mejora los límites inferiores de (1.0.17) y (1.0.18).

Nuestras pruebas reducen el análisis de $\lambda_1(\gamma)$ al estudio de la primer autovalor de un problema unidimensiónal de Sturm-Liouville con una degeneración en el origen. Bajo un cambio de variables, este problema se reduce a una ecuación de Bessel y nuestros resultados se obtienen mediante un análisis cuidadoso de la misma.

2. **Potenciales multipolares**

En la Sección 2.4 probamos desigualdades óptimas del tipo Hardy en todo el espacio $\mathbb{R}^N$, $N \geq 3$, para operadores de Schrödinger con varias singularidades cuadráticas de las formas

$$B_\lambda := -\Delta - \lambda \sum_{1 \leq i < j \leq n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2}, \quad \lambda > 0.$$

Más precisamente, mostramos que, independientemente de la ubicación de las singularidades $x_i \in \mathbb{R}^N$, B_λ es no-negativo en el sentido de formas cuadráticas en $L^2(\mathbb{R}^N)$, si y solo si $\lambda \leq (N-2)^2/n^2$, es decir,

$$\forall u \in H^1(\mathbb{R}^N), \quad \int |\nabla u|^2 dx \geq \frac{(N-2)^2}{n^2} \sum_{1\leq i<j\leq n} \frac{|x_i - x_j|^2}{|x-x_i|^2|x-x_j|^2} u^2 dx. \tag{1.0.22}$$

En otras palabras, esta desigualdad es óptima, es decir

$$\mu(\mathbb{R}^N) = \frac{(N-2)^2}{n^2}. \tag{1.0.23}$$

Esto tiene por objeto complementar la siguiente desigualdad, porbada por Bosi et al. [19]:

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx + \frac{(N-2)^2}{4n^2} \sum_{1\leq i<j\leq n} \int \frac{|x_i-x_j|^2}{|x-x_i|^2|x-x_j|^2} u^2 dx, \tag{1.0.24}$$

la cual es válida para cualquier $u \in H^1(\mathbb{R}^N)$, y un conjunto de polos $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$, $n \geq 2$.

La desigualdad (1.0.24) se obtuvo mediante el uso del método de "expansión de los cuadrados" en [19]. Al elegir los parámetros convenientes, podemos aplicar este método para obtener el resultado óptimo (1.0.23).

Además, los autores en [19] probaron desigualdades óptimas de Hardy para operadores de Schrödinger con un número finito de polos singulares del tipo

$$C_\lambda := -\Delta - \sum_{i=1}^{n} \frac{\lambda}{|x-x_i|^2},$$

dependiendo de restos de orden inferior en L^2, es decir,

$$u \in H^1(\mathbb{R}^N), \quad C \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{u^2}{|x-x_i|^2} dx \geq 0, \tag{1.0.25}$$

para cualquier $\mu \leq (N-2)^2/4$, y una constante $C > 0$ en función de N, n y $d := \min_{i\neq j} |x_i - x_j|/2$. Usando el resultado óptimo (1.0.23) obtenido para A_λ, construimos también ejemplos de dominios acotados Ω, para los cuales el término de orden inferior en (1.0.25) se puede despreciar. Las mejores límites inferiores se obtienen en el caso de potenciales bipolares, donde las singularidades x_i, $i \in \{1,2\}$ se encuentran en la frontera del dominio, es decir

$$\forall u \in H_0^1(\Omega), \quad \int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \Big[\frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}\Big] u^2 dx.$$

De esta manera se obtiene nuevas cotas inferiores para la constante óptima en la desigualdad multipolar de Hardy estándar para el operador C_λ en dominios acotados. Para más detalles de estos resultados nos referimos al Teorema 2.4.1 y las Proposiciones 2.4.1, 2.4.2.

Los resultados de este apartado se encuentran en [40].

3. **Número infinito de singularidades**

En la Sección 2.5 obtenemos cotas inferiores para la constante óptima en la desigualdad de Hardy correspondiente a una clase de operadores de Schrödinger con un número infinito de singularidades. Más precisamente, se estudia la positividad de los operadores de la forma

$$C_\lambda := -\Delta - \sum_{i\in\mathbb{Z}} \frac{\lambda}{|x-x_i|^2} \tag{1.0.26}$$

en dominios cilindricos, siempre y cuando los polos x_i se encuentran periódicamente en el eje central del cilindro $\mathbb{R}^N$, $n \geq 3$.

Capítulo 3: Controlabilidad de frontera para la ecuación de ondas con un potencial cuadrático localizado en el borde

El problema de la controlabilidad de un sistema de evolución implica analizar si, en un tiempo finito dado $T > 0$, la solución correspondiente puede ser dirigida a un estado final deseado, a partir de los datos iniciales perteneciendo a un determinado espacio funcional. Con el fin de controlar el sistema es necesario aplicar una fuerza externa adecuada (el control) para conducir todas las trayectorias del sistema a una configuración final. En general, el control puede actuar sobre un subconjunto de la frontera (control de frontera) o que podrán ser distribuido en un subconjunto abierto del dominio (control distribuido o interno), etc ..

El, por ahora el método clásico, el método de Hilbert (HUM), introducido por J. L. Lions [87], representa una herramienta sistemática en la Teoría del control para una amplia clase de ecuaciones de evolución. En términos generales, se afirma que el problema de la controlabilidad para un sistema de evolución es equivalente a la llamada desigualdad de observabilidad para el sistema homogéneo adjunto.

Existen diferentes métodos para probar la observabilidad. Entre ellos recordar las estimaciones de Carleman para ecuaciones parabólicas desarrolladas por Fursikov y Imanuvilov [69], análisis microlocal por Bardos, Lebeau y Rauch ([13], [12]), y también técnicas de multiplicadores, con los trabajos pioneros de Komornik y Zuazua ([82], [83], [114]) que se ajustan muy bien a los sistemas conservativos hiperbólicos.

Vamos a plantear el problema que nos interesa en el Capítulo 3. Por $N \geq 1$ se considera un dominio acotado regular $\Omega \subset \mathbb{R}^N$, donde $\Gamma = \partial\Omega$ denota su frontera. Por otra parte, se denota por Γ_0 una parte no vacía de Γ donde el control actúa, y consideramos el sistema de ondas

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{1.0.27}$$

Decimos que el sistema de la evolución (1.0.27) es exactamente controlable desde Γ_0, en algun tiempo $T > 0$, si para cualesquiera datos iniciales $(u_0, u_1) \in L^2(\Omega) \times H_\lambda'$ y cualquier objetivo $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda'$, existe un control $h \in L^2((0,T) \times \Gamma_0)$ de tal manera que la solución de (1.0.27) satisface:

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \text{para todo } x \in \Omega,$$

donde H_λ es el espacio inducido por el funcional de Hardy que fue definido por Vázquez y Zuazua [111] como el cierre de $C_c^\infty(\Omega)$ en la norma cuadrática

$$||u||_{H_\lambda}^2 := \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \frac{u^2}{|x|^2} dx, \tag{1.0.28}$$

y H_λ' denota su dual.

Este problema de controlabilidad fue analizado por Vancostenoble y Zuazua [109], bajo el presuponiendo que la singularidad $x = 0$ se encuentra en el interior de Ω. Ellos demostraron que el sistema (1.0.27) esta bien condicionado y es controlable para cualquier $\lambda \leq \lambda_\star := (N-2)^2/4$, donde la región de observabilidad Γ_0 viene dada por

$$\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}. \tag{1.0.29}$$

En el Capítulo 3, tratamos la controlabilidad del mismo en el caso de singularidades ubicadas en la frontera. Mostramos que, para la misma configuración geométrica como en [109], podemos aumentar el rango de valores de λ (desde $\lambda_\star$ a $\lambda(N) := N^2/4$) para los cuales la controlabilidad exacta del sistema (1.0.27) se cumple. Esta mejora en el resultado es consecuencia de las nuevas desigualdades de Hardy descritas en el Capítulo 2.

Demostramos con rigor la validez de la identidad de Pohozaev correspondiente al operador elíptico $A_\lambda := -\Delta - \lambda/|x|^2$. Como se muestra en el Capítulo 5, la regularidad elíptica clasica no se aplica a A_λ, y por tanto, ya que la singularidad se encuentra en el borde, esto afecta a la regularidad de la derivada normal de las funciones del dominio $D(A_\lambda)$ de A_λ, es decir,

$$D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}. \tag{1.0.30}$$

La principal dificultad en la demostración de la identidad de Pohozaev es precisamente debido a esta falta de regularidad. Una vez que esta dificultad técnica ha sido superada, la identidad de Pohozaev proporciona una directa herramienta para mostrar la desigualdad de observabilidad y el uso de la identidad de multiplicadores y de los argumentos de compacidad-unicidad de [88] (véase el Capítulo 3). Además, con el fin de obtener el tiempo óptimo de control que se espera $T > 2\max_{x\in\overline{\Omega}}|x|$ probamos una desigualdad óptima del tipo Hardy para obtener cotas superiores para el término $||x\cdot\nabla u||_{L^2(\Omega)}$ que aparece cuando se aplica el método de los multiplicadores. Estas son las principales herramientas utilizadas para demostrar la observabilidad de nuestro sistema.

Por último, en el Capítulo 3, Subsección 3.4.2, consideramos el sistema del tipo Schrödinger

$$\begin{cases} iu_t - \Delta u - \lambda\frac{u}{|x|^2} = 0, & (t,x)\in(0,T)\times\Omega,\\ u(t,x) = h(t,x), & (t,x)\in(0,T)\times\Gamma_0,\\ u(t,x) = 0, & (t,x)\in(0,T)\times(\Gamma\setminus\Gamma_0),\\ u(0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{1.0.31}$$

en donde la singularidad se encuentra en la frontera, y extendemos los resultados en [109] al caso de la singularidad de frontera en el nuevo rango de $\lambda \leq N^2/4$.

Los resultados del Capítulo 3 se han anunciado en [37]. La versión completa de los resultados en [37] se dan en [36].

Capítulo 4: Controlabilidad nula para la ecuación del calor con un potencial singular en la frontera

En el Capítulo 4 estudiamos el problema de control para la ecuación de calor con un potencial singular cuadrático $-\mu/|x|^2$.

Recientemente se han estudiado de manera intensa problemas parabólicos con potenciales singulares o degenerados. Entre ellos, recordamos los trabajos pioneros de Baras y Goldstein [9] en las que los autores consideran la ecuación del calor con la singularidad localizada en el interior del dominio. El trabajo mencionado proporciona condiciones necesarias y suficientes para para que el sistema corespondiente sea bien condicionado. Más precisamente, los autores en [9] probaron que el problema está bien puesto para cualquier $\mu \leq (N-2)^2/4$, mientras que si $\mu > (N-2)^2/4$ el sístema explota. Recordamos que el valor crítico $(N-2)^2/4$ es la mejor constante en la desigualdad de Hardy con una singularidad interna [79], [78].

Posteriormente, la comunidad matemática que trabaja en la teoria de control, ha estudiado los potenciales singulares. Los autores en Vancostenoble-Zuazua [108] mostraron que la ecuación de calor corespondiente esta controlada con un control rodeando la singularidad. Este resultado se ha generalizado en Ervedoza [54] por eliminar todas las restricciones geométricas para la región de control. Recientemente, [54] ha sido ligeramente complementado por Vancostenoble [107] en el estudio de algunas aplicaciones a los problemas inversos. En todos los casos anteriores, los autores demostraron que las propiedades de control y estabilización tienen mucho que ver con la desigualdad clásica de Hardy en la que la mejor constante es $(N-2)^2/4$.

En el Capítulo 4 se considera la ecuación del calor con el potencial $-\mu/|x|^2$, con la singularidad $x = 0$ colocada en la frontera del dominio. Este trabajo esté enfocado a extender el articulo [54] al caso de singularidades en el borde.

Nuestros resultados no se pueden deducir del caso de la singularidad interna y requiere un análisis independiente. La demostración está basada en las estimaciones de Carleman que son, por ahora, la herramienta clásica para probar las propiedades de observabilidad para los sistemas parabólicos. La mayor dificultad consiste en la búsqueda de funciones de peso adecuadas para el desarrollo de estimaciones de Carleman eficientes. En nuestro caso, los pesos utilizados en [108], [109], en el caso de singularidades internas, ni siquiera permite recuperar los resultados en el intervalo de parámetros $\mu \leq (N-2)^2/4$. Una buena adaptación de los pesos en [54], [69] se presenta con todos los detalles en el Capítulo 4.

Vamos a exponer brevemente nuestras principales aportaciones. Sea $N \geq 2$ y consideramos un dominio regular acotado $\Omega \subset \mathbb{R}^N$ tal que $0 \in \partial\Omega$ y que $\omega \subset \Omega$ sea un conjunto abierto no vacío. Supongamos también que $T > 0$ es fijo. Estamos interesados en la cuestión de controlabilidad de el siguiente problema

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.0.32}$$

donde $u_0 \in L^2(\Omega)$ y $f \in L^2(\Omega \times (0,T))$ es una función con el soporte en la región de control ω.

El problema de la controlabilidad nula se plantea de la siguiente manera: Dada cualquier $u_0 \in L^2(\Omega)$, se debe buscar una función $f \in L^2(\omega \times (0,T))$ tal que la solución de (1.0.32) satisface

$$u(x,T) = 0, \qquad x \in \Omega. \tag{1.0.33}$$

Extendemos los resultados del control en [54] al caso de la singularidad en la frontera de la siguiente manera. Ampliamos la gama de valores μ de $\mu \leq (N-2)^2/4$ a $\mu \leq N^2/4$ para el cual el sistema (1.0.32) se puede controlar con un control distribuido en un subconjunto abierto $\omega \subset \Omega$ para las dos siguientes configuraciones geométricas (véase la Figura 1.3):

1. Si los puntos de la frontera de $x \in \partial\Omega$ satisfacen la condicion $x \cdot \nu \leq 0$ en una vecindad del origen, entonces la región de control ω es un subconjunto abierto de Ω
2. En caso contrario, la región de control ω necesita rodear la singularidad sin que la contenga.

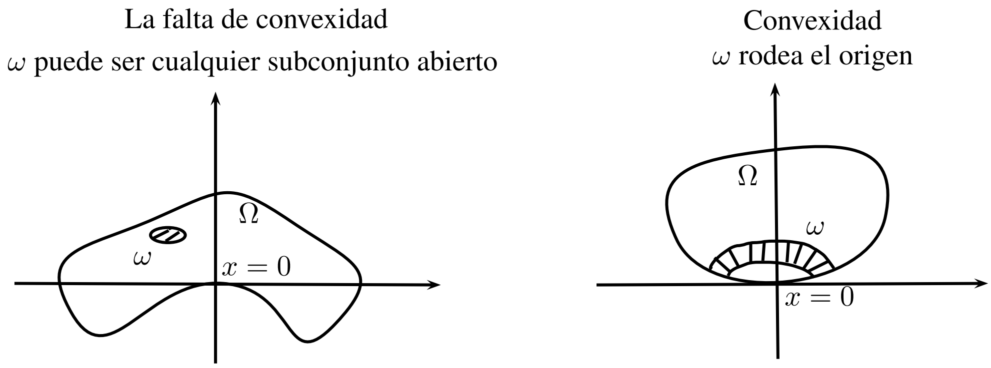

Figure 1.3: Limitaciones para la región de control

La cuestión de que el resultado que obtenemos de los controles internos que rodean el origen, como en la Figura 1.3, sea óptimo sigue siendo un problema abierto.

Los resultados del Capítulo 4 se incluyen en [35].

Capítulo 5: La regularidad del operador de Schrödinger con potencial singular cuadrático

Consideremos un dominio acotado regular $\Omega \subset \mathbb{R}^N$, $N \geq 3$, que contiene el origen (excepto en el caso $N = 1$ en la que $x = 0$ se encuentra en el borde), y la fuerza $f \in L^2(\Omega)$. Consideramos también el problema elíptico asociado al operador de Schrödinger $A_\lambda = -\Delta - \lambda/|x|^2$, que es

$$\begin{cases} -\Delta u - \lambda \frac{u}{|x|^2} = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.0.34}$$

La formulación variacional (1.0.34) tiene sentido en el espacio de Hilbert H_λ, con $\lambda \leq \lambda_\star = (N-2)^2/4$, es decir:

$$\begin{cases} a(u,v) = (f,v)_{L^2(\Omega)}, & \forall v \in H_\lambda, \\ u \in H_\lambda, \end{cases} \tag{1.0.35}$$

donde $a(\cdot,\cdot) : H_\lambda \times H_\lambda \to \mathbb{R}$ está definido por

$$a(u,v) = \int_\Omega \Big(\nabla u \cdot \nabla v - \lambda \frac{uv}{|x|^2}\Big) dx. \tag{1.0.36}$$

y constituye una forma bilineal, continua y coerciva en H_λ.

Debido al lema de Lax-Milgram y a la desigualdad óptima de Hardy (1.0.2), el problema (1.0.34) admite una única solución variacional $u \in H_\lambda$ de (1.0.35), con $\lambda \leq \lambda_\star$. En otras palabras, existe un homeomorfismo de $L^2(\Omega)$ a $D(A_\lambda)$ donde $D(A_\lambda)$ denota el dominio de A_λ que se define en (1.0.30).

Para cualquier $N \geq 3$, se ha calculado el espectro discreto de A_λ de manera explícita en [111] y el comportamiento asintótico de la parte radial de la primera autofunción, la más singular, está dada por

$$\phi_1(x) \sim |x|^{-\frac{N-2}{2}+\sqrt{\lambda_\star-\lambda}}, \quad \text{cuando } x \sim 0.$$

Esto sugiere la falta de regularidad elíptica estándar en el polo singular para las distribuciónes de $D(A_\lambda)$. Tal fenómeno ocurre también en el caso del Laplaciano clásico cuando se consideran problemas en dominios no regulares (por ejemplo, los dominios con esquinas o grietas) (véase por ejemplo Grisvard [76]).

En el Capítulo 5 estudiamos la regularidad de la solución del problema (1.0.34). Demostramos que, debido a la presencia de la singularidad en el origen, existe una falta de regularidad elíptica estándar para las distribuciones en $D(A_\lambda)$. En primer lugar, damos resultados óptimos de regularidad en cualquier dimensión en el caso radial, a tráves de los espacios de Sobolev fraccionarios (Ω es una bola y $f(x) = f(|x|)$ es una función radial) siendo capaces de determinar una fórmula explícita para la solución de (1.0.34). Nuestras herramientas se basan en una transformación logarítmica y el método de las variaciones de las constantes. En primer lugar, analizamos el problema en una dimensión. Luego mostramos que el problema multidimensional radial se reduce a la unidimensional.

En el caso de soluciones no-radiales, probamos el comportamiento asintótico en el origen de los $u \in D(A_\lambda)$ cuando λ se está acercando a la $\lambda_\star$ desde abajo. En la prueba se utilizan técnicas espectrales y propiedades finas de las funciones de Bessel. Paralelamente, durante la redacción de esta memoria, se han obtenido resultados asymptoticós similares en [62] mediante diferentes técnicas. El lector interasado puede consultar y comparar las tecnicas usadas. Los resultados del Capítulo 5 se utilizan en el análisis de las aproximaciones numéricas del Capítulo 6.

Capítulo 6: FEM Adaptativo para el operador de Schrödinger A_λ

El análisis y la implementacion de métodos numéricos eficientes de aproximación para problemas relacionados con los potenciales singulares, constituyen una parte importante de esta tesis.

Hay varios métodos numéricos utilizados por la comunidad matemática con el fin de aproximar la solución de una EDO específica (por ejemplo, Runge-Kutta) o EDP (por ejemplo, diferencias finitas, volúmenes finitos, etc ..).

A partir de la obra de Courant [44], seguido pocos años después por Pólya [95], [96] y Synge [102], el Método de Elementos Finitos (FEM) se utiliza de manera generalizada para resolver ecuaciones en derivadas parciales.

En el Capítulo 6 se discuten, precisamente el rendimiento de los sistemas basados en elementos finitos P1 lineales a trozos y continuos, para la resolución del problema elíptico singular

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.0.37}$$

Consideramos el rango de los parámetros $\lambda \leq \lambda_\star = (N-2)^2/4$ dada por las desigualdades óptimas de Hardy. Vamos a suponer que el lado derecho f está en $L^2(\Omega)$ y $\Omega \subset \mathbb{R}^N$, $N \geq 1$ es un dominio que contiene la singularidad $x = 0$ en el interior (excepto en el caso unidimensional, en el que la singularidad surge en el borde).

El objetivo de este capítulo consiste en la construcción de FEM adaptados para el problema (1.0.37) para optimizar la velocidad de convergencia en la norma H_λ.

Debido a la presencia de la singularidad, de acuerdo con el Capítulo 5, la regularidad elíptica estándar falla para (1.0.37) y por lo tanto, el FEM clásico con mallas uniformes, ofrece peores tasas de convergencia que los que da el Laplaciano (que corresponde a $\lambda = 0$). La regularidad de la solución de (1.0.37) baja desde $H^2(\Omega)$ (en el caso de $\lambda = 0$) a una cierta $H^{1+s}(\Omega)$ con $0 < s < 1$, como se muestra en el Capítulo 5. en este caso al aplicar FEM estándar, la tasa de convergencia disminuye de h a h^s. Esto se muestra en la Subsección **??** por medio de la interpolación de Clement [43].

En el FEM estándar con una malla uniforme de tamaño $h > 0$, usando $n(h) \sim h^{-N}$ elementos, se obtiene un error de orden h^s en lugar de h, ya que se produce para el Laplaciano clásico.

Esto significa que, con el fin de aproximar la solución de (1.0.37) dentro de un error prescrito $h > 0$, nos vemos obligado a proceder con una malla más fina de la que se requiere para los operadores estándar elípticos donde la regularidad de las soluciones es H^2.

El FEM adaptativo (AFEM) ha sido creado para compensar la falta de regularidad en el análisis del coste computacional, mediante el uso de aproximaciones finitas de los elementos sobre mallas no uniformes. Muchas veces en la práctica, este método tiene un coste computacional óptimo (véase por ejemplo [33], [91]) que se basa en un análisis a posteriori, que implica solamente la solución discreta y los datos.

Por lo tanto, en la Subsección 6.2.1 podemos mejorar las tasas de convergencia mediante el uso de mallas no uniformes que se adaptan a la singularidad del potencial. Para el problema unidimensional, se utilizan estimaciones a priori para mostrar las tasas óptimas de convergencia, es decir, se obtiene un error prescrito $h > 0$ en la norma H_λ, $\lambda < \lambda_\star$ mediante el uso de $n(h) \sim h^{-1}$ nodos o grados de libertad (DOF abv.).

En el caso bidimensional se considera el problema con singularidades colocadas en la frontera ya que el problema continuo, no está bien planteado si la singularidad está en el interior. Por simplicidad se considera $\Omega = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$, con la singularidad $x = 0$.

Bajo las suposiciones anteriores, en la Subsección 6.3.4, se muestran y discuten algunos experimentos numéricos basados en un análisis a posteriori para el problema bidimensional (1.0.37) correspondiente a A_λ y $\Omega = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$.

Hemos desarrollado y empleado un código numérico que es una adaptación del código tutorial de Chen-Zhang [42] para nuestra específica EDP. El código en [42] se ha desarrollado utilizando estimacíones de error a posteriori con el fin de optimizar el coste computacional de elementos finitos para el Laplaciano en un dominio en forma de L. Los resultados numéricos se realizaron para $f = 1$ en (1.0.37). Dado que la singularidad se encuentra en el borde, el valor crítico λ es $\lambda_\star = 1$. Debido a estos hechos, desarrollamos simulaciones para varios valores de λ: $\lambda = -0.5$, $\lambda = 0$, $\lambda = 0.5$, $\lambda = 1$, $\lambda = 1.1$.

Los experimentos numéricos que realizamos dan lugar a las siguientes conclusiones:

1. Para $\lambda = -0,5$ la solución numérica a (1.0.37) no requiere un refinamiento particular de la malla en la vecindad de la singularidad $x = 0$.

2. El caso $\lambda = 0$ corresponde con al caso del operador de Laplace en el que la tasa de convergencia óptima se obtiene para mallas uniformes.

3. Para los casos $\lambda = 0,5$ y $\lambda = 1$ el problema (1.0.37) está bien planteado y se observa que la solución numérica correspondiente requiere mallas con matices más finas en la singularidad $x = 0$. Esto es debido a la falta de regularidad del problema continuo. En el caso crítico $\lambda = 1$ el refinamiento es aún más fino que en el caso subcrítico $\lambda = 0,5$ ya que la solución correspondiente es menos regular que en el primer caso.

4. En el caso supercrítico $\lambda = 1.1$ observamos que la solución numérica explota después de

un número finito de iteraciones. Esto está de acuerdo con el hecho de que el problema esta mal condicionado para $\lambda > 1$.

Chapter 1

Introduction

Motivation and bibliographical comments

In the theory of singular PDEs, the Hardy inequality has been playing a crucial role when analyzing and describing the qualitative properties of Schrödinger operators with inverse-square potentials.

In 1925, G. H. Hardy [79] proved that, in the one dimensional case, it holds

$$\forall u \in H_0^1(0,\infty), \qquad \int_0^\infty |u'(r)|^2 dr \geq \frac{1}{4}\int_0^\infty \frac{u^2(r)}{r^2} dr. \tag{1.0.1}$$

Few years later, inequality (1.0.1) was also mentioned in [Leray [85], page 204] where the author studied the existence of regular solutions for unlimited viscous fluids. Later on, (1.0.1) was generalized in Hardy-Littlewood-Pólya [78] to the multi-dimensional case. The so-called classical Hardy inequality asserts that, for any Ω an open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin, and for any $u \in H_0^1(\Omega)$ then $u/|x| \in L^2(\Omega)$ and the following estimate holds:

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} dx. \tag{1.0.2}$$

The constant $(N-2)^2/4$ in (1.0.2) is optimal and not attained in $H_0^1(\Omega)$. In other words, the continuous embedding $H_0^1(\Omega) \subset \L^2(\Omega; |x|^{-2}dx)$ is not compact.

The study of Hardy inequality (1.0.2) is motivated by applications to various fields. In particular, inequality (1.0.2) is known by quantum physicists to be closely related to the Uncertainty Principle of Heisenberg (see e.g. Fefferman [61]). Besides, it can be applied both to obtain lower bounds for the spectrum of (generalized) Schrödinger operators of the form $-\Delta - V(x)$ (for potentials $V(x) > 0$ with isolated singularities, $V \in L^1_{loc}(\Omega)$) with Dirichlet boundary conditions, and to analyze the boundary behavior of heat kernels (see Davis [45]).

The Hardy inequality plays also an important role in Quantum Mechanics when studying the non-relativistic Schrödinger equation for a single particle moving in an electric field:

$$Eu = -\frac{\hbar^2}{2\mu}\Delta u - \frac{e^2}{4\pi\varepsilon_0 r}u. \tag{1.0.3}$$

This form of the Schrödinger equation applies to the Hydrogen atom (see e.g. [6]), where E is a proportionality constant, $\hbar$ is the Planck's constant, e is the electron charge, $r = |x|$ is the position, wherein ε_0 is the electric constant (permittivity of free space) and $\mu = m_e m_p/(m_e + m_p)$ is the 2-body reduced mass of the Hydrogen nucleus (just a proton) of mass m_p and the electron of mass m_e.

The Hardy potential $V = |x|^{-2}$ in (1.0.2) has homogeneity equal to -2 which is critical from both mathematical and physical view points. Because of that, they cannot be considered as lower order perturbations of the Laplacian.

In addition, potentials with critical singularities also apply in Quantum Cosmological models as emphasized by the Wheeler-de-Witt equation (see e.g. [14]), or in the linearization of nonlinear PDEs which describe the features of solutions for models arising in combustion theory (e.g. Gelfand [71]).

Let us motivate these facts by considering the semi-linear elliptic PDE

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.0.4}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is an open bounded domain. The nonlinearity f is assumed to be continuous, positive, increasing and convex function so that $f(0) > 0$ and

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty. \tag{1.0.5}$$

The equation (1.0.4) appears when studying the description of a ball of isothermal gas in gravitational equilibrium proposed by Lord Kelvin [41].

It is well-known from Brezis-Cazenave [22] and Brezis-Nirenberg [25] that there exists a finite positive number $\lambda^\star = \lambda^\star(\Omega)$ called the *extremal value* such that, for any $0 \leq \lambda < \lambda^\star$, problem (1.0.4) has a classical solution $\underline{u}_\lambda \in C^2(\overline{\Omega})$ which is *minimal* among all possible solutions. Otherwise, if $\lambda > \lambda^\star$, the Dirichlet problem (1.0.4) has no weak solutions (not even in $L^1(\Omega)$).

Then Brezis and Vázquez proved in [26] that, under the condition (1.0.5) on f, the following limit exists a.e.

$$u^\star(x) := \lim_{\lambda\to\lambda^\star} \underline{u}_\lambda(x) \tag{1.0.6}$$

and $u^\star \in L^1(\Omega)$ is a week solution of problem (1.0.4), the so-called *extremal solution* corresponding to $\lambda^\star$. Since the properties of classical extremal solutions had been well studied, the authors in [26] were interested on the characterization of the unbounded extremal solutions $u^\star$ (in the energy class $H_0^1(\Omega)$) and the extremal value $\lambda_\star$.

In Brezis-Vázquez [26] it was shown that $u^\star \in H_0^1(\Omega)$ is an unbounded extremal solution for (1.0.4) corresponding to $\lambda = \lambda^\star$ if and only if the first eigenvalue of the linearized operator $-\Delta - \lambda^\star f'(u^\star)$ is non-negative, that is

$$\int_\Omega |\nabla u|^2 dx \geq \lambda^\star \int_\Omega f'(u^\star) u^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{1.0.7}$$

In particular, if $f(u) = e^u$, this corresponds to the celebrated Gelfand problem [72]. If moreover $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$, then we can explicitly deduce that $(\lambda^\star, u^\star) = (2(N-2), -2\log(|x|))$ for any $N \geq 10$, since the inequality

$$\int_\Omega |\nabla u|^2 dx \geq 2(N-2) \int_\Omega \frac{u^2}{|x|^2} dx \tag{1.0.8}$$

is true, due to the Hardy inequality (1.0.2). If $N \leq 9$ it was shown in [26] that there are not extremal solutions for the Gelfand problem [72].

Another typical example for f in (1.0.4) is $f(u) = (1+u)^p$, $p > 1$ (see e.g. [75], [98]).

For a general overview on the existing literature on the Hardy inequality we recommend the review papers by Davies [46] and Brezis-Marcus [23].

The reader interested on the extensions of the classical Hardy inequality (1.0.2) is referred, in particular, to the following papers and the references therein: [26], [71], [4], [27], [10], [65], [105], [111], [104], [84], [48], [51], [50], [66].

Recently, improved versions of (1.0.1)-(1.0.2) have been established in open bounded domains containing the origin (see [26], [3], [60]). Generalizations of the Hardy inequality to the case of the $p-$Laplacian have been recently obtained in [58]. We also mention the papers [19], [64] and the references therein for the analysis of multipolar singularities. There has been also an intensive research on singular potentials involving the distance to the boundary (e.g. [23], [24]).

Content of the Thesis

In Chapter 2 of this Thesis we study Hardy type inequalities for Schrödinger operators with quadratic singular potentials, where the singularities are located either in the interior or on the boundary of an open (bounded or unbounded, smooth or non-smooth) domain. We analyze the cases of potentials with one, various or infinite number of singular poles.

These results on Hardy inequalities are further used in the next Chapters of the Thesis in applications to the control of the heat and wave equations with singular potentials or to design efficient numerical methods for elliptic problems with quadratic singularities.

Thus, in Chapters 3 and 4 we analyze the control properties of evolution equations (both hyperbolic and parabolic) with one singular potential localized on the boundary of a smooth domain and more precisely, we emphasize how the singularity affects the well-posedness, controllability aspects, etc...

In the context of the controllability problem for the wave equation with boundary singular potential in Chapter 3, we need to prove an observability inequality for the solution of the adjoint system in which, the total energy is recovered by measuring the normal derivative on the boundary, in an weighted L^2-norm with a degeneracy at the origin. For that, we first justify some hidden regularity property for the normal derivative of the solution to the adjoint system with initial data in the energy space.

This is not a direct consequence of the well-posedness property in the energy space but it is deserved to the intrinsic properties of the wave equation and the presence of the singular potential at the boundary.

In Chapter 5 we study the regularity of the Schrödinger operator with quadratic singular potentials. We first analyze the one dimensional case with the singularity located on the boundary and later the multi-dimensional case with internal singularity.

Then, in Chapter 6 we end up with the development and analysis of an adaptive numerical approximation method of such singular PDEs, in the context of Finite Element Methods (FEM).

Since critical singularities affect the regularity of the solutions of the considered PDEs, the classical elliptic theory does not apply and, consequently, as we will see, it cannot be translated neither in the continuous context nor for numerical approximation methods.

The main results of this Thesis can be classified as follows:

1. Chapter 2: Hardy inequalities

    1). Quadratic singular potential localized on the boundary: we prove lower bounds for optimal Hardy constants in conical domains which improve some well-known results in the literature, and we show optimal inequalities in smooth domains which provide better results than the classical Hardy inequality (1.0.2).

2). Multipolar potentials: we determine optimal constants for inequalities with potentials having a finite number of quadratic singularities, independent on the location of the poles. Our results answer to the question of optimality of some inequalities discussed in Bosi-Dolbeault-Esteban [19].

3). Potentials with an infinite number of singularities: we determine non-trivial lower bounds to inequalities in strip-like domains for potentials with infinite number of singularities located periodically on a line. To the best of our knowledge, these results are completely new.

2. Chapter 3: Boundary controllability for the wave and Schrödinger equations with singular potentials on the boundary

   We prove boundary controllability results for the wave and Schrödinger equations with boundary singular potentials. We extend the analysis in Vaconstenoble-Zuazua [109] in the context of interior singularities to the boundary singularity case. Our results yield to show finer observability inequality in order to compensate the lack of regularity at the boundary.

3. Chapter 4: Null-controllability of the heat equation with singular potentials on the boundary

   We prove null-controllability results with internal controls for the heat equation with boundary singular potential. Using Carleman estimates, we reprove and extend similar results in Ervedoza [54] in the case of interior singularity to the one with boundary singularity. Our results are based on a proper modification of the weights of the Carleman estimates in [54].

4. Chapter 5: Schrödinger operators with Dirichlet boundary conditions and interior isolated singularities

   We prove the regularity of radial solutions and determine the asymptotic behavior for the radial part of the non-radial solutions of the problem under consideration. To the best of our knowledge, our analysis is new.

5. Chapter 6: Numerical approximations for Schrödinger operators with inverse-square potentials

   FE approximations for the one dimensional problem are proved by means of an a priori analysis. Moreover, we present numerical experiments which concern the Adaptive FEM

(AFEM) and the a posteriori analysis for the two dimensional one with boundary singular potential. Our numerical results confirm the theoretical aspects developed in this Thesis.

In the sequel we briefly describe in more details our main results and the techniques we have developed.

Chapter 2: Hardy inequalities

In Chapter 2 of this Thesis we analyze the positivity (in L^2-quadratic forms) of Schrödinger operators $-\Delta - V(x)$, $V(x) > 0$, with inverse square potentials of the form

$$A_\lambda := -\Delta - \frac{\lambda}{|x|^2}, \tag{1.0.9}$$

$$B_\lambda := -\Delta - \lambda \sum_{1\leq i<j\leq n} \frac{|x_i - x_j|^2}{|x-x_i|^2|x-x_j|^2}, \tag{1.0.10}$$

and,

$$C_\lambda := -\Delta - \sum_{i\in I} \frac{a_i}{|x-x_i|^2}, \tag{1.0.11}$$

where $\lambda > 0$, $n \in \mathbb{N}$ and $N \geq 1$ are integers and I is a countable set of indices such that $a_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$ with $i \in I$. These potentials have one, a finite or an infinite number of singular poles.

More precisely, we are interested in the optimal Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \mu(\Omega) \int_\Omega V(x) u^2 dx, \quad \forall u \in H_0^1(\Omega), \tag{1.0.12}$$

where Ω is an open subset of $\mathbb{R}^N$, $N \geq 1$, and the optimal constant $\mu(\Omega)$ stands for

$$\mu(\Omega) := \inf \Big\{ \int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega V(x)u^2 dx,\ u \in H_0^1(\Omega) \Big\}.$$

Of course, the problem becomes interesting when the poles of the potentials V in (1.0.9)-(1.0.11) belong to $\overline{\Omega}$; otherwise it is a regular perturbation of the Laplacian.

In Chapter 2 we address the question of determining the value $\mu(\Omega)$ (or alternatively to show upper and lower bounds), in terms of the nature of the potential V given in each one of the situations (1.0.9)-(1.0.11).

Next we briefly describe the main results of Chapter 2.

1. **Boundary singularities.**

   In Section 2.3 we discuss the positivity of singular operators of the form

$$A_\lambda = -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda > 0,$$

in which the singularity $x = 0$ is placed on the boundary $\partial\Omega$ of an open domain $\Omega \subset \mathbb{R}^N$ and $N \geq 2$. We analyze two types of geometrical configurations Ω depending on the smoothness at the origin.

- *Smooth domains (or Lipschitz domains of class C^2 near $x = 0$).* Roughly speaking, we show that, the value $\mu(\Omega)$ jumps from $(N-2)^2/4$, the classical constant in the case of Hardy inequality on domains with interior singularity, to the new Hardy constant $N^2/4$. This was first achieved for $\Omega = \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is the half-space of $\mathbb{R}^N$ in which the condition $x_N > 0$ holds. More precisely, for any $N \geq 2$, Filippas, Tertikas and Tidblom proved in [67] the new optimal Hardy inequality

$$\forall u \in H^1_0(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{u^2}{|x|^2} dx. \tag{1.0.13}$$

In Section 2.3.1 we prove inequality (1.0.13) in any smooth bounded domain Ω containing the origin on the boundary. These results turn out to be closely related to the global convexity of Ω but independent of the geometry of Ω at the origin $x = 0$ (as emphasized in Figure 1.1).

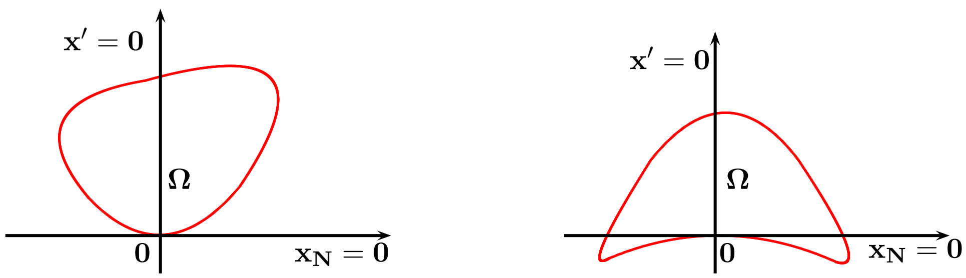

Figure 1.1: Convex geometry (on the left); Lack of convexity (on the right).

More precisely, independently of the geometry of Ω, there exists a constant $r_0 > 0$ depending on Ω such that

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}, \tag{1.0.14}$$

where $B_{r_0}(0) \subset \mathbb{R}^N$ is the ball of radius r_0 centered at the origin. As a consequence $\mu(\Omega) \leq N^2/4$. In particular, if Ω is convex it holds that $\mu(\Omega) = N^2/4$. Otherwise, we point out that, for some particular geometries of Ω, an extra lower order term (in the L^2-norm) is needed for the inequality to hold:

$$\forall u \in H_0^1(\Omega), \quad C\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx, \tag{1.0.15}$$

for some positive constant $C > 0$ depending on Ω. There are non-convex domains for which the lower order term in (1.0.15) can be removed but there are counterexamples, as well (see Proposition 2.3.4).

In addition, in bounded domains, we improve these results by addressing the question of finding nontrivial lower bounds for the difference

$$I[u] = \int_\Omega |\nabla u|^2 dx - \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx,$$

which are the so-called Hardy-Poincaré type inequalities.

Part of these results were first announced in Cazacu and Zuazua [39] and Cazacu [38]. Similar results were obtained independently by Fall and Musina [59], [57]. In Section 2.3.1 we present our contributions complementing those in [59], [57], and also giving some simpler or alternative proofs.

- *Non-smooth domains (conical domains with the vertex located at $x = 0$).* Another interesting situation is that in which the boundary develops corners at the singularity. The most relevant example is that of a conical domain with the vertex at the origin $x = 0$ (See Figure 1.2).

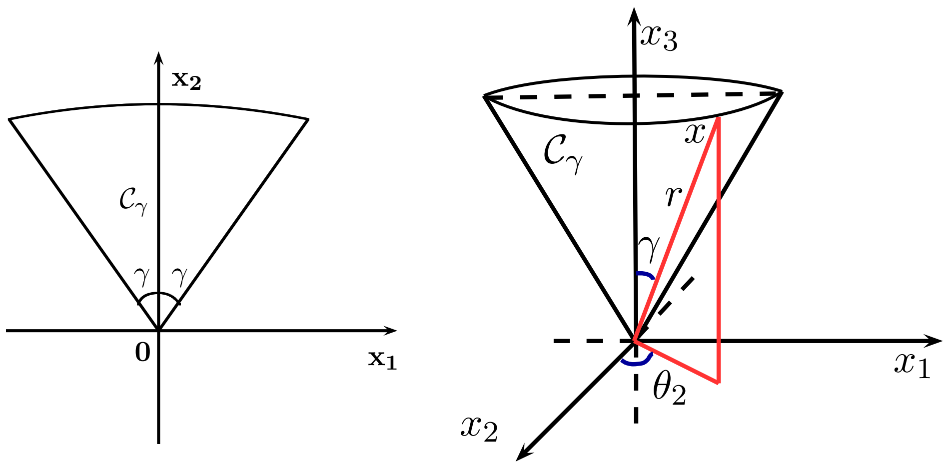

Figure 1.2: The bidimensional sector with the aperture γ (on the left). The three-dimensional cone with the aperture γ (on the right).

We consider conical domains $\Omega = \mathcal{C}_\gamma \subset \mathbb{R}^N$, $N \geq 2$, with the vertex at $x = 0$ and amplitude $\gamma \in (0, \pi)$ (see Figure 1.2).

It is well-known that (see e.g. Tintarev [93])

$$\mu(\mathcal{C}_\gamma) = \frac{(N-2)^2}{4} + \lambda_1(\gamma), \tag{1.0.16}$$

where $\lambda_1(\gamma)$ is the first eigenvalue of the Laplace-Beltrami operator on the the surface $\mathcal{C}_\gamma \cap S^{N-1}$, S^{N-1} denoting the unit sphere in $\mathbb{R}^N$ with $(N-1)$-Haussdorff measure.

The question of studying the exact value of $\mu(\Omega)$ in conical domains has been addressed in a number of cases (see e.g. [15], [30]). If $N = 2$ then $\mu(\mathcal{C}_\gamma) = \pi^2/(2\gamma)^2$. If $N = 4$ then $\mu(\mathcal{C}_\gamma) = \pi^2/\gamma^2$. If $N = 3$ or $N \geq 5$ the value $\lambda_1(\gamma)$ in (1.0.16) has not been determined explicitly. Despite of that, lower and upper bounds have been proved.

To the best of our knowledge, the best lower bounds have been proved in [30]. In particular, if $N = 3$ and $\gamma \in (0, \pi/2)$ then (cf. [94], [30])

$$\lambda_1(\gamma) \geq \frac{j_0^2}{\gamma^2} - \Big(\frac{1}{2} - \frac{1}{\pi^2}\Big) \tag{1.0.17}$$

where j_0 denotes the first zero of the Bessel function J_0. Also, if $N \geq 5$ and $\gamma \in (0, \pi/2)$ then (cf. [30])

$$\lambda_1(\gamma) \geq \frac{j^2_{(N-3)/2}}{\gamma^2} - \frac{(N-2)(N-1)}{6} \tag{1.0.18}$$

where $j_{(N-3)/2}$ is the first positive zero of the Bessel function $J_{(N-3)/2}$.

In Section 2.3.2, for any dimension $N \geq 2$, we prove that

$$\lambda_1(\gamma) \geq (N-1)\frac{\pi^2}{4\gamma^2}. \tag{1.0.19}$$

In view of (1.0.19), our result improves the lower bounds (1.0.17)-(1.0.18) in [30].

In addition, in Section 2.3.2 we give an alternative proof for the asymptotic value of $\lambda_1(\gamma)$ in [30], as γ tends to zero, that is

$$\lim_{\gamma \to 0} \lambda_1(\gamma)\gamma^2 = j^2_{(N-3)/2}, \quad \forall N \geq 3. \tag{1.0.20}$$

In particular, in any dimension $N \geq 3$, we prove the lower bound

$$\lambda_1(\gamma) \geq \Big(\frac{\sin\gamma}{\gamma}\Big)^{N-2} \frac{j^2_{(N-3)/2}}{\gamma^2}, \quad \forall \gamma \in (0, \pi), \tag{1.0.21}$$

which, for small γ, improves the lower bound in (1.0.17)-(1.0.18) as well.

Our proofs reduce the analysis of $\lambda_1(\gamma)$ to the study of the first eigenvalue of a one-dimensional Sturm-Liouville problem with degeneracy at the origin. Under a change of variables this problem is reduced to a Bessel equation and our results are obtained by a careful analysis of it.

2. **Multipolar potentials**

In Section 2.4 we prove optimal Hardy-type inequalities in the whole space $\mathbb{R}^N$, $N \geq 3$, for Schrödinger operators with positive multi-singular inverse square potentials of the form

$$B_\lambda := -\Delta - \lambda \sum_{1 \leq i < j \leq n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2}, \quad \lambda > 0.$$

More precisely, we show that, independently of the location of the singularities $x_i \in \mathbb{R}^N$, B_λ is non-negative in the sense of L^2 quadratic forms in $\mathbb{R}^N$, if and only if $\lambda \leq (N-2)^2/n^2$, i.e.

$$\forall u \in H^1(\mathbb{R}^N), \quad \int |\nabla u|^2 dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 dx. \tag{1.0.22}$$

In other words, this inequality is optimal, that is

$$\mu(\mathbb{R}^N) = \frac{(N-2)^2}{n^2}. \tag{1.0.23}$$

This aims to complement the following inequality due to Bosi et al. [19]:

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^n \int \frac{u^2}{|x - x_i|^2} dx + \frac{(N-2)^2}{4n^2} \sum_{1 \leq i < j \leq n} \int \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 dx, \tag{1.0.24}$$

which holds for any $u \in H^1(\mathbb{R}^N)$ and any set of poles $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$, $n \geq 2$.

Inequality (1.0.24) was obtained by using the "expansion of the square" method in [19]. By choosing convenient parameters, we apply this method to get the optimal result (1.0.23).

In addition, the authors in [19] proved optimal Hardy inequalities for Schrödinger operators with a finite number of singular poles of the type

$$C_\lambda := -\Delta - \sum_{i=1}^n \frac{\lambda}{|x - x_i|^2},$$

up to lower order L^2-reminder terms, that is

$$u \in H^1(\mathbb{R}^N), \quad C \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{u^2}{|x - x_i|^2} dx \geq 0, \tag{1.0.25}$$

for any $\mu \leq (N-2)^2/4$ and some constant $C > 0$ depending on N, n and $d := \min_{i \neq j} |x_i - x_j|/2$. Using the optimal result (1.0.23) obtained for A_λ, we also construct examples of bounded domains Ω, in which the lower order term in (1.0.25) can be removed. The best lower bounds are obtained in the case of bipolar potentials, where the singularities x_i, $i \in \{1, 2\}$ are located on the boundary of the domain, i.e.

$$\forall u \in H_0^1(\Omega), \qquad \int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \Big[\frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}\Big] u^2 dx.$$

In this way we obtain new lower bounds for the optimal constant in the standard multi-singular Hardy inequality for the operator C_λ in bounded domains. For more explicit statements of these results we refer to Theorem 2.4.1 and Propositions 2.4.1, 2.4.2.

The results of this paragraph are contained in [40].

3. **Infinite number of singularities**

In Section 2.5 we obtain lower bounds for the optimal constant in the Hardy inequality corresponding to a class of Schrödinger operators with an infinite number of singularities. More precisely, we study the positivity of operators of the form

$$C_\lambda := -\Delta - \sum_{i \in \mathbb{Z}} \frac{\lambda}{|x-x_i|^2} \tag{1.0.26}$$

in strip-like domains, provided the poles x_i are located periodically on a straight line of $\mathbb{R}^N$, $N \geq 3$.

Chapter 3: Boundary controllability of the wave equation with a quadratic potential localized on the boundary

The problem of controllability of an evolution system implies to analyze whether in a given finite time $T > 0$, the corresponding solution can be leaded to a desired final state, starting from any initial data belonging to a given functional space. In order to control the system it is necessary to apply a proper external force (the control) to lead all the trajectories of the system to a final configuration. In general, the control could act on a subset of the boundary (boundary control) or it may be distributed in an open subset of the domain (distributed or internal control), etc...

The, by now classical, Hilbert Uniqueness Method (HUM) introduced by J. L. Lions [87], represents a systematic tool in Control Theory for a large class of evolution equations.

Roughly speaking, it states that the controllability problem for an evolution system is equivalent to the so-called observability inequality for the homogeneous adjoint system.

There are different methods to prove observability. Among them we recall the Carleman estimates for parabolic equations developed by Fursikov and Imanuvilov [69], microlocal analysis due to Bardos, Lebeau and Rauch ([13] , [12]), but also multiplier techniques with the pioneering papers by Komornik and Zuazua ([82], [83], [114]) which fit very well to conservative hyperbolic systems.

Let us now state the problem we are interested in Chapter 3. For $N \geq 1$ we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $\Gamma = \partial\Omega$ denoting its boundary. Moreover, we denote by Γ_0 a non-empty part of the set Γ on which the control acts and we consider the wave-like system

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{1.0.27}$$

We say that the evolution system (1.0.27) is exactly controllable from Γ_0, in some $T > 0$, if for any initial data $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and any target $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$, there exists a control $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of (1.0.27) satisfies:

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \text{for all } x \in \Omega,$$

where H_λ is the space induced by the Hardy functional defined in Vázquez and Zuazua [111] as the closure of $C_c^\infty(\Omega)$ in the quadratic norm

$$||u||_{H_\lambda}^2 := \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \frac{u^2}{|x|^2} dx, \tag{1.0.28}$$

and $H_\lambda^{'}$ denotes its dual.

This controllability problem was analyzed by Vancostenoble and Zuazua [109] under the assumption that the singularity $x = 0$ is located in the interior of Ω. They proved well-posedness and exact controllability of system (1.0.27) for any $\lambda \leq \lambda_\star := (N-2)^2/4$, where the boundary observability region Γ_0 is given by

$$\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}. \tag{1.0.29}$$

In Chapter 3, we address the same controllability question in the case of boundary singularities. More precisely, we show that, for the same geometrical setup as in [109], we can increase the range of values λ (from $\lambda_\star$ to $\lambda(N) := N^2/4$) for which the exact boundary controllability of system (1.0.27) holds. This improvement is a consequence of the new Hardy inequalities in Chapter 2.

We show rigorously the validity of the Pohozaev identity corresponding to the elliptic operator $A_\lambda = -\Delta - \lambda/|x|^2$. As shown in Chapter 5 standard elliptic regularity does not apply to A_λ, and therefore, since the singularity is located on the boundary, this affects the regularity of the normal derivative of the functions in the domain $D(A_\lambda)$ of A_λ, that is

$$D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}. \tag{1.0.30}$$

The main difficulty in the proof of the Pohozaev identity is precisely due to this lack of regularity. Once this technical difficulty has been overcome, the Pohozaev identity provides a direct tool to show observability by showing the multiplier identity and using compactness-uniqueness arguments as the ones in [88] (see Chapter 3). In addition, in order to obtain an optimal expected control time $T > 2\max_{x\in\overline{\Omega}}|x|$ we prove a sharp Hardy-type inequality to get upper bounds for the term $||x \cdot \nabla u||_{L^2(\Omega)}$ appearing when applying the method of multipliers. These are the main tools used to prove observability of our system.

Finally, in Chapter 3, Subsection 3.4.2, we consider the Schrödinger-like system

$$\begin{cases} iu_t - \Delta u - \lambda\frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.0.31}$$

where the singularity is located on the boundary, and we extend the results in [109] to the case of boundary singularity in the new range $\lambda \leq N^2/4$.

The results of Chapter 3 have been announced in [37]. The complete version of the results in [37] are given in [36].

Chapter 4: Internal null-controllability of the heat equation with boundary singular potential

In Chapter 4 we are concerned with the null-controllability problem for the heat equation with quadratic singular potential $-\mu/|x|^2$.

Parabolic evolution problems with the potential $-\mu/|x|^2$ have been intensively studied in the recent past. Among them, we remind the pioneering work by Baras and Goldstein [9] in which the authors considered the heat equation with the singularity localized in the interior of the domain. They derived necessary and sufficient conditions for the well-posedness of such systems. More precisely, they showed that the well-posedness holds true whenever $\mu \leq (N-2)^2/4$, whereas if $\mu > (N-2)^2/4$ there is instantaneous blow-up. Remark that the critical value $(N-2)^2/4$ is the best constant in the corresponding Hardy inequality with internal singularity [79], [78]. Later on, the issue of singular potentials has been also analyzed by the control community. The authors in Vancostenoble-Zuazua [108] showed that the corresponding heat equation can be controlled by a distributed control which sourrounds the singularity. This result has been generalized in Ervedoza [54] removing all geometric constraints on the control region. Recently, [54] has been slightly complemented by Vancostenoble [107] when studying some applications to inverse problems. In all situations above the authors showed that the well-posedness, control and stabilization properties are very much related to the classical Hardy inequality in which the best constant is $(N-2)^2/4$.

In Chapter 4 we consider the heat equation with the potential $-\mu/|x|^2$, where the singularity $x = 0$ is located on the boundary of the domain. This work is aimed to extend to the case of boundary singularities the paper [54].

Our results cannot be deduced from the case of interior singularity and requires an independent analysis. Our proof is based on the Carleman estimates which are by now the classical tool to prove observability properties for parabolic systems. The major difficulty consists in finding proper weight functions to develop efficient Carleman estimates. In our case, the weights used in [108], [109], for interior singularities, do not even allow to recover the results in the range of parameters $\mu \leq (N-2)^2/4$. A proper adaptation of the weights in [54],[69] is presented with full details in Chapter 4.

Let us next briefly state our main contributions. Let $N \geq 2$ and consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$, and let $\omega \subset \Omega$ be a non-empty open set. Assume also that $T > 0$ is fixed. We are interested in the question of controllability of the following problem

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.0.32}$$

where $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega \times (0,T))$ is a function supported in the control region ω.

The null-controllability problem reads as follows: Given any $u_0 \in L^2(\Omega)$, to find a function $f \in L^2(\omega \times (0,T))$ such that the solution of (1.0.32) satisfies

$$u(x,T) = 0, \qquad x \in \Omega. \tag{1.0.33}$$

We extend the control results in [54] to the case of boundary singularity as follows. We enlarge the range of values μ from $\mu \leq (N-2)^2/4$ to $\mu \leq N^2/4$ for which the system (1.0.32) is controllable with a control distributed in an open subset $\omega \subset \Omega$ in the following two geometrical configurations (see Figure 1.3):

1. If the points on the boundary $x \in \partial\Omega$ satisfy the conditions $x \cdot \nu \leq 0$ in a neighborhood of the origin, then the control region ω is any open subset of Ω
2. Otherwise, the control region ω needs to surround the singularity without containing it.

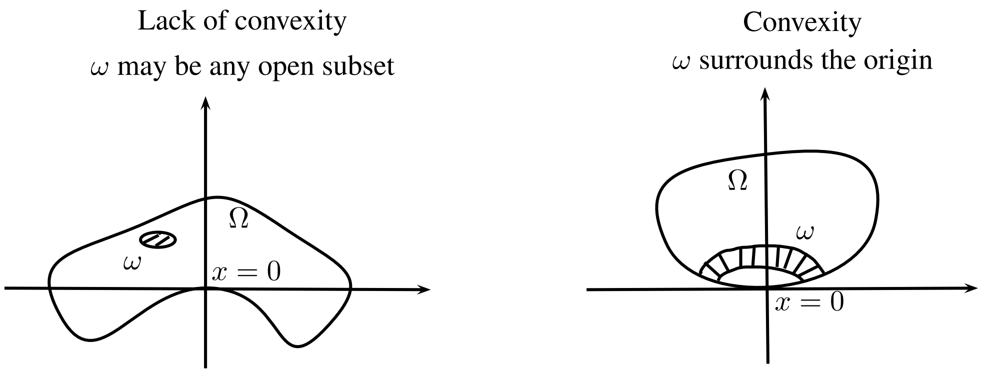

Figure 1.3: Limitations for the control region

The question of whether the result we obtain for internal controls surrounding the origin, as in Figure 1.3, is optimal is still an open problem.

The results of Chapter 4 are included in [35].

Chapter 5: Regularity of the Schrödinger operator with quadratic singular potential

Let us consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, containing the origin (except for the case $N=1$ in which $x=0$ is located on the boundary), and a force term $f \in L^2(\Omega)$. We consider also the elliptic problem associated to the Schrödinger operator $A_\lambda = -\Delta - \lambda/|x|^2$, that is

$$\begin{cases} -\Delta u - \lambda \frac{u}{|x|^2} = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.0.34}$$

The variational formulation of (1.0.34) makes sense in the Hilbert space H_λ, with $\lambda \leq \lambda_\star = (N-2)^2/4$ and is as follows:

$$\begin{cases} a(u,v) = (f,v)_{L^2(\Omega)}, & \forall v \in H_\lambda, \\ u \in H_\lambda, \end{cases} \tag{1.0.35}$$

where $a(\cdot,\cdot) : H_\lambda \times H_\lambda \to \mathbb{R}$ is defined by

$$a(u,v) = \int_\Omega \Big[\nabla u \cdot \nabla v - \lambda \frac{uv}{|x|^2}\Big] dx. \tag{1.0.36}$$

and constitutes a bilinear, continuous and coercive form in H_λ.

Due to the Lax-Milgram lemma and the optimal Hardy inequality (1.0.2), the problem (1.0.34) admits a unique variational solution $u \in H_\lambda$ of (1.0.35), with $\lambda \leq \lambda_\star$. In other words, there exists a homeomorphism from $L^2(\Omega)$ to $D(A_\lambda)$ where $D(A_\lambda)$ denotes the domain of A_λ defined in (1.0.30).

For any $N \geq 3$, the discrete spectrum of A_λ has been explicitly computed in [111] and the asymptotic behavior of the radial part of the first eigenfunction, the most singular one, is given by

$$\phi_1(x) \sim |x|^{-\frac{N-2}{2}+\sqrt{\lambda_\star-\lambda}}, \quad \text{as } x \sim 0.$$

This suggests the lack of standard elliptic regularity for distributions in $D(A_\lambda)$ at the singular pole. Such a phenomenon appears also for the classical Laplacian in non-smooth domains (e.g. domains with corners or cracks) (see e.g. Grisvard [76]).

In Chapter 5 we study the regularity of the solution to problem (1.0.34). We will prove that, due to the presence of the singularity at the origin, there is a lack of standard elliptic regularity for distributions in $D(A_\lambda)$. We firstly give optimal regularity results in any dimension in the radial case, by means of fractional Sobolev spaces (Ω is a ball and $f(x) = f(|x|)$ is a radial function) since we are able to determine an explicit formula for the solution of (1.0.34). Our tools are based on a logarithmic transformation and the method of variations of constants. We firstly analyze the one-dimensional problem. Then we show that the radial multi-dimensional problem reduces to the one-dimensional one.

In the case of non-radial solutions, we prove the asymptotic behavior at the origin for $u \in D(A_\lambda)$ when λ is approaching $\lambda_\star$ from below. In the proof we use spectral techniques and fine properties of Bessel functions. The results of Chapter 5 will be used in the analysis of the numerical approximations of Chapter 6.

Chapter 6: Adaptive FEM approximations for the Schrödinger operator A_λ

The analysis and implementation of efficient numerical approximation methods for problems involving singular potentials, constitute another an important part of this Thesis.

There are various numerical methods used by the mathematical community in order to approximate the solution of a specific ODE (e.g. Runge-Kutta method) or PDE (e.g. finite differences, finite volumes, etc...).

Starting with the work by Courant [44], followed few years later by Pólya [95], [96] and Synge [102], the Finite Element Method (FEM) is now widely used to solve PDEs.

in Chapter 6 we discuss precisely the performance of the finite element schemes based on P1 piecewise linear and continuous elements, when solving the singular elliptic problem

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.0.37}$$

We consider the range of parameters $\lambda \leq \lambda_\star = (N-2)^2/4$ given by optimal Hardy inequalities. We will assume that the right hand side f belongs to $L^2(\Omega)$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a domain containing the singularity $x = 0$ in the interior (except in the one dimensional case, in which the singularity arises at the boundary).

The aim of this chapter consists in constructing adapted FE methods for problem (1.0.37) to optimize the rate of convergence in the H_λ-norm.

Due to the presence of the singularity, according to Chapter 5, the standard elliptic regularity fails for (1.0.37) and therefore, the classical FEM on uniform meshes, provides worse convergence rates than those for the Laplacian (which corresponds to $\lambda = 0$). The regularity of the solution of (1.0.37) drops down from $H^2(\Omega)$ (in the case $\lambda = 0$) to a certain $H^{1+s}(\Omega)$ with $0 < s < 1$ as shown in Chapter 5. In this case when applying standard FEM, the rate of convergence diminishes from h to h^s. This is shown in Subsection 6.2.2 by means of the Clement interpolation [43].

By the standard FEM on a unifom mesh of size $h > 0$, by using $n(h) \sim h^{-N}$ nodes, we obtain an error of order h^s instead of h as it occurs for the classical Laplacian.

This means that, in order to approximate the solution of (1.0.37) within a prescribed error $h > 0$, we are forced to proceed with a finer mesh that is required for standard elliptic operators for which the regularity of the solutions is H^2.

The Adaptive FEM (AFEM) has been introduced to compensate the lack of regularity in the analysis of the computational cost, by using finite element approximations on non-uniform meshes.

In practice, this method has an optimal computational cost (cf. [53], [91]) and relies on an a posteriori analysis involving only the discrete solution and the data.

Hence, in Section 6.2.1 we improve the convergence rates by using non-uniform meshes which are adapted to the singularity of the potential. For the one dimensional problem, we use a priori estimates to show optimal convergence rates, i.e., we get a prescribed error $h > 0$ in the H_λ norm, $\lambda < \lambda_\star$ by using $n(h) \sim h^{-1}$ elements or degrees of freedom (abv. DoF).

In the bi-dimensional case we consider the problem of boundary singularities since the continuous problem is not well-posed when the singularity is in interior. For simplicity we consider $\Omega = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$ with the singularity $x = 0$.

Under the assumptions above, in Section 6.3.4, we show and discuss some numerical experiments based on an a posteriori analysis for the bi-dimensional problem (1.0.37) corresponding to A_λ and $\Omega = [-1, 1] \times [0, 1] \subset \mathbb{R}^2$.

We developed and employed a numerical code which is an adaptation of the tutorial code Chen-Zhang [42] to our specific PDE. The code in [42] was developed using a posteriori error estimates in order to optimize the computational cost of FEM for the Laplacian in an L-shaped domain. Our numerical results correspond to the case $f = 1$ in (1.0.37). Since the singularity is located on the boundary, the critical value λ for the well-posedness of (1.0.37) is $\lambda_\star = 1$. Due to that we develop simulations for several values of λ: $\lambda = -0.5$, $\lambda = 0$, $\lambda = 0.5$, $\lambda = 1$, $\lambda = 1.1$.

The analysis of the numerical experiments yields the following conclusions:

1. For $\lambda = -0.5$ the numerical solution to (1.0.37) does not require a particular mesh refinement in the neighborhood of the singularity $x = 0$.

2. The case $\lambda = 0$ corresponds to the Laplace operator in which the optimal convergence rate is obtained for uniform meshes.

3. In the cases $\lambda = 0.5$ and $\lambda = 1$ the problem (1.0.37) is well-posed and we observe that the corresponding numerical solution requires meshes with finer refinements at the singularity $x = 0$. This is because the lack of regularity of the continuous problem. In the critical case $\lambda = 1$ the refinement is still finer than for the subcritical one $\lambda = 0.5$ since the corresponding solution is less regular in the first case.

4. In the supercritical case $\lambda = 1.1$ we observe that the numerical solution breaks down after a finite number of iterations. This is in accordance with the ill-posedness of the problem for $\lambda > 1$.

Chapter 2

Hardy inequalities

Abstract. In this chapter we discuss Hardy-type inequalities for operators with a finite or an infinite number of quadratic singularities. We determine either optimal constants or lower bounds for such inequalities regardless of the location of the singularities involved in the problem. Roughly speaking, we show non-trivial inequalities in (smooth or non-smooth) domains containing the singularities either in interior or on the boundary, improving some results already existing in the literature.

2.1. Introduction

Motivation. Hardy inequalities represent a classical subject in which there has been intensive research in the recent past, mainly motivated by its applications to Partial Differential Equations (PDE's). In this chapter we are focused on inequalities related to Schrödinger operators $-\Delta + V(x)$ with inverse square potentials V

The singularity of inverse square potentials cannot be considered as a lower perturbation of the Laplacian since it has homogeneity -2, being critical from both a mathematical and a physical viewpoint.

Potentials of type $1/|x|^2$ arise, for instance, in Frank et al. [68] where a classification of singular spherical potentials is given in terms of the limit $\lim_{r\to 0} r^2 V(r)$. When the limit is finite and non-trivial, V is said to be a *transition potential*. This potential also arises in point-dipole interactions in molecular physics (see Lévy-Leblond [86]), where the interaction among the poles depends on their relative partitions and the intensity of the singularity in each of them.

Multipolar potentials of type $V = \sum_{i=1}^{n} \alpha_i/|x - x_i|^2$ are associated with the interaction of a finite number of electric dipoles. They describe molecular systems consisting of n nuclei of unit charge located at a finite number of points $x_1, \dots, x_n$ and of n electrons. This type of systems are described by the Hartree-Fock model, where Coulomb multi-singular potentials arise in correspondence to the interactions between the electrons and the fixed nuclei, see Catto et al. [34].

Classical Hardy inequality. The positivity and coercivity (in the L^2 norm) of such operators above are strongly related to Hardy-type inequalities. The first well-known result relies on a 1-d inequality due to G. H. Hardy [79] which claims that

$$\forall u \in H_0^1(0,\infty), \quad \int_0^\infty u_x^2 dx > \frac{1}{4}\int_0^\infty \frac{u^2}{x^2}dx, \tag{2.1.1}$$

where the constant $1/4$ is optimal and not attained in $H_0^1(0,\infty)$. More precisely,

$$\inf_{u\in H_0^1(0,\infty)} \Big(\int_0^\infty |u_x|^2 dx \Big/ \int_0^\infty \frac{u^2}{x^2}dx\Big) = \frac{1}{4}. \tag{2.1.2}$$

A minimizing sequence of (2.1.2) is not unique and may be chosen in different ways. For instance, let us consider the index parameter $\varepsilon > 0$ aimed to be small. Then the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ defined by

$$u_\varepsilon(x) = \begin{cases} x^{1/2+\varepsilon}, & 0 < x \leq 1, \\ x^{1/2-\varepsilon}, & 1 < x, \end{cases} \tag{2.1.3}$$

minimizes the quotient of (2.1.2) when ε tends to zero. More precisely we have

$$\int_0^\infty (u_\varepsilon)_x^2 dx \Big/ \int_0^\infty \frac{u_\varepsilon^2}{x^2}dx = \frac{1}{4} + \varepsilon^2 \to \frac{1}{4}, \qquad \text{as } \varepsilon \to 0. \tag{2.1.4}$$

Later on, this inequality was generalized to the multi-d case by Hardy-Littlewood-Polya [78] showing that for any Ω an open subset of $\mathbb{R}^N$, $N \geq 3$ containing the origin, it holds that

$$\forall u \in H_0^1(\Omega), \quad \int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx, \tag{2.1.5}$$

and the constant $(N-2)^2/4$ is optimal and not attained. For $N = 2$, inequality (1.0.2) is trivially true. A minimizing sequence for the optimal constant may be given by

$$u_\varepsilon(x) = \begin{cases} |x|^{-(N-2)/2+\varepsilon}, & 0 < |x| \leq 1, \\ |x|^{-(N-2)/2-\varepsilon}, & 1 < |x|. \end{cases} \tag{2.1.6}$$

The reader interested in the existing literature on the extensions of the classical Hardy inequality (2.1.5) is referred, in particular, to the following papers and the references therein: [26], [71], [4], [27], [10], [65], [105], [111], [104], [84], [48], [51], [50], [66].

Recently, improved versions of (2.1.1)-(2.1.5) have been established in open bounded domains containing the origin (see [26], [3]). We also mention the papers [19], [64] and the references therein for discussing inequalities with multipolar singularities. There has been also an intensive research for singular potentials involving the distance to the boundary (e.g. [23], [24]).

Main goal. To fix the ideas, let us consider Ω to be an open subset of $\mathbb{R}^N$, $N \geq 2$. For a general quadratic singular potential V (placing the role of each singular potential of the Schrödinger operators defined in (1.0.9)-(1.0.11)) we call $\mu(\Omega)$ to be the best constant in the corresponding Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \mu(\Omega)\int_\Omega V(x)u^2 dx, \quad \forall \quad u \in H_0^1(\Omega), \tag{2.1.7}$$

i.e.

$$\mu(\Omega) := \inf \Big\{ \int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega V(x) u^2 dx,\ u \in H_0^1(\Omega) \Big\}. \tag{2.1.8}$$

In this Chapter we are aimed to study the value of $\mu(\Omega)$. In the worst cases we only show upper and lower bounds for $\mu(\Omega)$.

State of the art and description of the main results

1. **Boundary singularities.**
In Section 2.3 we study qualitative properties of singular operators of the form

$$A_\lambda = -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda > 0,$$

in which the singularity $x = 0$ lies on the boundary, since this subject has been less investigated so far in the literature. To be more precise, throughout Section 2.3, we consider Ω to be a subset of $\mathbb{R}^N$ with the origin $x = 0$ placed on its boundary $\partial\Omega$. We are seeking for the value of

$$\mu(\Omega) = \inf \Big\{ \int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega \frac{u^2}{|x|^2} dx,\ u \in H_0^1(\Omega) \Big\}.$$

Of course, in view of (2.1.5), $\mu(\Omega) \geq (N-2)^2/4$. We address the question of finding non-trivial cases in which $\mu(\Omega) > (N-2)^2/4$. We analyze two types of domains regardless the regularity of the boundary at the singular pole.

- *Inequalities in smooth domains.* The authors in [59] showed that the strict inequality $\mu(\Omega) > (N-2)^2/4$ holds true when Ω is a bounded domain of class C^2. Actually, the value $\mu(\Omega)$ depends on the geometric properties of the boundary $\partial\Omega$ at the singularity. The first explicit example has been given for $\Omega = \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is the half-space of $\mathbb{R}^N$ in which the condition $x_N > 0$ holds. More precisely, for any $N \geq 1$, Filippas, Tertikas and Tidblom proved in [67] the new Hardy inequality:

$$\forall u \in H_0^1(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{u^2}{|x|^2} dx. \tag{2.1.9}$$

  Moreover, they proved the constant $N^2/4$ to be optimal (cf. Corollary 2.4, pp. 12, [67]) i.e. $\mu(\mathbb{R}^N_+) = N^2/4$. The constant is not attained. However, a minimizing sequence as is given by

$$\varepsilon > 0,\ u_\varepsilon = \begin{cases} x_N, & |x| \leq 1, x \in \mathbb{R}^N_+, \\ x_N |x|^{-N/2-\varepsilon}, & |x| > 1, x \in \mathbb{R}^N_+. \end{cases} \tag{2.1.10}$$

  In this section we extend inequality (2.1.9) to any smooth bounded domain Ω containing the origin on the boundary. We point out, that for some particular geometries of Ω, we have to pay the prize of adding a lower order term in L^2-norm in the gradient part of (2.1.9). These results turn out to be closely related to the convexity of Ω at the origin, but also to the global geometry of Ω as emphasized later on.

Part of the results in this section were first announced in [39], [38]. At the same time, the authors in [59], [57] have been working independently on the same topic getting very similar results. Because [59], [57] yield some similar results, we thus present here in a detailed manner the most novel aspects not addressed in [59]. We also present some alternative proofs.

- *Inequalities in non-smooth domains (conical domains with the vertex located at $x = 0$).* Another interesting situation appears in non-smooth domains Ω, when the boundary develops corners or cusps at the singularity. The most relevant example of such domain is represented by a conical domain with the vertex at the origin $x = 0$.

  The question of studying the exact value of $\mu(\Omega)$ in conical domains has been full-filled in bi-dimensional case. More precisely, if $\mathcal{C}_\gamma$ is the conical sector with the amplitude $\gamma \in (0, \pi)$, then (e.g. [29]) $\mu(\mathcal{C}_\gamma) = (\pi/2\gamma)^2$.

  On the other hand (cf. [15]), if $N = 4$ then $\mu(\mathcal{C}_\gamma) = \pi^2/\gamma^2$. Otherwise, if $N \geq 3$ and $N \neq 4$, to the best of our knowledge, the constant $\mu(\mathcal{C}_\gamma)$ has not been explicitly determined so far. However, lower bounds have been proved as emphasized in the Introduction of this memoir. in Section 2.3.2 we prove lower bounds for $\mu(\mathcal{C}_\gamma)$, when $\mathcal{C}_\gamma$ is a conical domain in dimensions $N \geq 3$, with the vertex in $x = 0$ with amplitude $\gamma \in (0, \pi)$. In particular, we improve the lower bounds in [30]. In addition, we give an alternative proof for the asymptotic value of $\mu(\mathcal{C}_\gamma)$ in [30], as γ tends to zero.

2. **Various singularities**

In this Section we prove optimal Hardy-type inequalities for Schrödinger operators with positive multi-singular inverse square potentials of the form

$$A_\lambda := -\Delta - \lambda \sum_{1\leq i<j\leq n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2}, \quad \lambda > 0.$$

More precisely, we show that A_λ is non-negative in the sense of L^2 quadratic forms in $\mathbb{R}^N$, if and only if $\lambda \leq (N-2)^2/n^2$, independently of the number n and location of the singularities $x_i \in \mathbb{R}^N$, where $N \geq 3$ denotes the space dimension. This aims to complement some of the results in Bosi et al. [19] obtained by the "expansion of the square" method. Due to the interaction of poles, our optimal result provides a singular quadratic potential behaving like $(n-1)(N-2)^2/(2n^2|x-x_i|^2)$ at each pole x_i. Besides, the authors in [19] showed optimal Hardy inequalities for Schrödinger operators with a finite number of singular poles of the type $B_\lambda := -\Delta - \sum_{i=1}^{n} \lambda/|x - x_i|^2$, up to lower order L^2-reminder terms. By means of the optimal results obtained for A_λ, we also build some examples of bounded domains Ω in which these lower order terms can be removed in $H_0^1(\Omega)$. In this way we obtain new lower bounds for the optimal constant in the standard multi-singular Hardy inequality for the operator B_λ in bounded domains. The best lower bounds are obtained when the singularities x_i are located on the boundary of the domain.

3. **Infinite number of singularities**

2.2. General strategy to prove Hardy-type inequalities

There are several techniques for proving Hardy inequalities in smooth domains (including the whole space) which are all interlinked by the following integral identity.

Proposition 2.2.1. *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an open set and let $x_1, \ldots, x_n \in \overline{\Omega}$. We also consider a function $\phi : \Omega \to \overline{\mathbb{R}}$ such that $\phi(x) \neq 0$ in $\Omega \setminus \{x_1, \ldots, x_n\}$ and $\phi \in C^\infty(\Omega \setminus \{x_1, x_2, \ldots x_n\})$. Then it holds that*

$$\int_\Omega \Big[|\nabla u|^2 + \frac{\Delta\phi}{\phi}u^2\Big]dx = \int_\Omega \Big|\nabla u - \frac{\nabla\phi}{\phi}u\Big|^2 dx = \int_\Omega \phi^2|\nabla(u\phi^{-1})|^2 dx, \tag{2.2.1}$$

for all $u \in C_0^1(\Omega \setminus \{x_1, x_2, \ldots x_n\})$.

The proof of (2.2.1) can be done using integration by parts as shown at the end of this section. In particular, (2.2.1) can be extended to test functions $u \in H_0^1(\Omega)$ since $C_0^1(\Omega \setminus \{x_1, \ldots, x_n\})$ is dense in $H_0^1(\Omega)$.

The identity (2.2.1) could be extended to more general classes of distributions ϕ depending on the applications that we have in mind. Here we are interested in applications to Hardy inequalities with singular or multipolar potentials located at the poles $x_1, \ldots, x_n$, with $x_i \neq x_j$ for all $i \neq j$, $i, j \in \{1, \ldots, n\}$, $n \geq 1$.

Various aspects of the identities involved in (2.2.1) have been used in the literature to prove and analyze Hardy inequalities in different contexts. But, as far as we know, (2.2.1) has not been stated explicitly as it stands before.

Identity (2.2.1) could be directly applied to obtain Hardy inequalities with potentials of the form $-\Delta\phi/\phi$, i.e.

$$\int_\Omega |\nabla u|^2 dx \geq \int_\Omega \Big(-\frac{\Delta\phi}{\phi}\Big)u^2 dx, \quad \forall u \in H_0^1(\Omega). \tag{2.2.2}$$

In order to derive inequalities for a concrete potential $V = V(x) \in L^1_{loc}(\Omega)$, one needs to look for a corresponding ϕ such that

$$-\frac{\Delta\phi}{\phi} \geq V(x), \quad \forall x \in \Omega. \tag{2.2.3}$$

Some of the existing techniques to prove Hardy-type inequalities use "the expansion of the square" method (e.g. [19]) or suitable functional transformations (e.g. [26], [5]). In view of (2.2.1), all these techniques are actually equivalent and the problem can always be reduced to checking pointwise inequalities for a potential V and a corresponding ϕ as in (2.2.3).

Optimality. For a general ϕ satisfying (2.2.1), we cannot say anything about the optimality of (2.2.2). To argue in that sense, next we give a counterexample by means of the standard Hardy inequality. Assume $\Omega = \mathbb{R}^N$, $N \geq 3$, and let $\lambda < \lambda_\star := (N-2)^2/4$. Then we consider

$$\phi = |x|^{-(N-2)/2+\sqrt{\lambda_\star - \lambda}},$$

and observe that $\phi > 0$ in $\mathbb{R}^N$, $\phi \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and therefore ϕ satisfies the identity (2.2.1) before. Then for such ϕ, inequality (2.2.2) becomes

$$\int |\nabla u|^2 dx \geq \lambda \int \frac{u^2}{|x|^2} dx, \quad \forall u \in H^1(\mathbb{R}^N),$$

which is not optimal as follows from (2.1.5).

Proof of Proposition 2.2.1. For a given $u \in H_0^1(\Omega \setminus \{x_1, \dots, x_n\})$ we introduce the transformation $u = \phi v$. Then we get

$$|\nabla u|^2 = |\phi|^2 v^2 + \phi^2 |\nabla v|^2 + 2\nabla\phi \cdot \nabla v \phi v.$$

Applying integration by parts it follows

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx &= \int_\Omega |\nabla \phi|^2 v^2 dx + \int_\Omega \phi^2 |\nabla v|^2 dx + \frac{1}{2}\int_\Omega \nabla(\phi^2)\cdot\nabla(v^2) dx \\
&= \int_\Omega |\nabla \phi|^2 v^2 dx + \int_\Omega \phi^2 |\nabla v|^2 dx - \frac{1}{2}\int_\Omega v^2 \mathrm{div}(2\phi\nabla\phi) dx \\
&= \int_\Omega |\nabla \phi|^2 v^2 dx + \int_\Omega \phi^2 |\nabla \phi|^2 dx - \frac{1}{2}\int_\Omega v^2 (2\phi\Delta\phi + 2|\nabla\phi|^2) dx \\
&= \int_\Omega |\nabla v|^2 \phi^2 dx - \int_\Omega \phi\Delta\phi v^2 dx. \qquad (2.2.4)
\end{aligned}$$

Therefore we obtain

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla v|^2 \phi^2 dx - \int_\Omega \frac{\Delta\phi}{\phi} u^2 dx.$$

In conclusion we get (2.2.1). □

2.3. Hardy inequalities with boundary quadratic singular potentials

In this section we study the positivity (in the L^2-quadratic norm) of the singular operator

$$A_\lambda = -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda > 0,$$

when the origin $x = 0$ is located on the boundary of an open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$.

To begin with, we recall some general properties of the Hardy inequality corresponding to A_λ above, which are stated in Proposition 2.3.1 and are independent on the location of the singularity.

Proposition 2.3.1. *The following properties are satisfied.*

1) Anti-monotonicity. If $\Omega_1 \subset \Omega_2$ then

$$\mu(\Omega_1) \geq \mu(\Omega_2). \qquad (2.3.1)$$

2) Invariance under dilations. Regardless whether $0 \in \partial\Omega$ or not, it holds

$$\mu(\Omega) = \mu(\lambda\Omega), \quad \forall\lambda > 0. \qquad (2.3.2)$$

3) Invariance under rotations. If T is a rotation centered at 0, then

$$\mu(\Omega) = \mu(T(\Omega)) \qquad (2.3.3)$$

Proof of Proposition 2.3.1. 1). This can be easily seen by extending by zero the function in $H_0^1(\Omega_1)$ to the exterior of Ω_1.
2). Let us fix $u \in H_0^1(\lambda\Omega)$ and consider the transformation $v(x) := \lambda^{N-2/2}u(\lambda x)$ which belongs to $H_0^1(\Omega)$. Then we have

$$\int_{\Omega} |\nabla v|^2 dx = \int_{\lambda\Omega} |\nabla u(y)|^2 dy, \qquad \int_{\Omega} \frac{v^2(x)}{|x|^2} dx = \int_{\lambda\Omega} \frac{u^2(y)}{|y|^2} dy,$$

which completes the proof of (2.3.2)
3). Considering the transformation

$$v(y) = u(x) = u(Ty) = u(T_1(y), T_2(y), ..., T_N(y)),$$

we have

$$\frac{\partial u}{\partial x_i} = \sum_j \frac{\partial v}{\partial y_j}\frac{\partial y_j}{\partial x_i} = \sum_i \frac{\partial (u \circ T)}{\partial y_j}\frac{\partial T_j^{-1}}{\partial x_i}.$$

Therefore, due to the fact that the associated matrix of T is ortonormal we obtain

$$|\nabla u|^2 = \Big|\sum_j \frac{\partial (u\circ T)}{\partial y_j}\nabla T_j^{-1}\Big|^2 = \sum_j \Big|\frac{\partial (u\circ T)}{\partial y_j}\Big|^2 |\nabla T_j^{-1}|^2 = \sum_j \Big|\frac{\partial (u\circ T)}{\partial y_j}\Big|^2 = |\nabla (u\circ T)|^2.$$

From this, and due to the fact that a rotation is an isometry which has the jacobian equal to 1 we get

$$\int_{T(\Omega)} |\nabla u(x)|^2 dx = \int_{\Omega} |\nabla (u\circ T)(y)|^2 J(T) dy = \int_{\Omega} |\nabla (u\circ T)(y)|^2 dy$$

$$\int_{T(\Omega)} \frac{u^2(x)}{|x|^2} dx = \int_{\Omega} \frac{|u\circ T(y)|^2}{|T(y)|^2} J(T) dy = \int_{\Omega} \frac{|u\circ T|^2}{|y|^2} dy.$$

From above we obtain (2.3.3). □

2.3.1. Smooth domains

In the sequel we present some preliminary results on the optimal constant in the Hardy inequality which in particular rely on Proposition 2.3.1 and inequality (4.2.7).

Preliminary results

We assume $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a smooth domain such that $0 \in \partial\Omega$. Moreover we introduce the ball

$$B_\gamma = \{x = (x', x_N) \in \mathbb{R}^N_+ \mid |x'|^2 + (x_N - 1)^2 < \gamma^2\} \tag{2.3.4}$$

Proposition 2.3.2. *It holds that*

$$\mu(\mathbb{R}^N_+) = \mu(B_\gamma) = \frac{N^2}{4} \quad \forall \gamma > 0.$$

Proof. Indeed, $B_\gamma \subset \mathbb{R}^N_+$ and by anti-monotonicity (2.3.1) we deduce $\mu(B_\gamma) \geq \mu(\mathbb{R}^N_+)$ for all $\gamma > 0$. In consequence, $\mu(\mathbb{R}^N_+) \leq \inf_{\gamma>0} \mu(B_\gamma)$. Let be $\varepsilon > 0$. By the definition of $\mu(\mathbb{R}^N_+)$ there exists $u_\varepsilon \in C_c^\infty(\mathbb{R}^N_+)$ such that

$$\int_{\mathbb{R}^N_+} \frac{u_\varepsilon^2}{|x|^2} dx = 1, \quad \int_{\mathbb{R}^N_+} |\nabla u_\varepsilon|^2 dx \leq \mu(\mathbb{R}^N_+) + \varepsilon. \tag{2.3.5}$$

Then there exists $m_\varepsilon \in \mathbb{N}$ s.t. $u_\varepsilon \in C_c^\infty(B_{m_\varepsilon})$. Hence

$$\mu(B_{m_\varepsilon}) \leq \int_{\mathbb{R}^N_+} |\nabla u_\varepsilon|^2 dx \leq \mu(\mathbb{R}^N_+) + \varepsilon.$$

Passing to the limit with respect to ε we obtain

$$\inf_{\gamma>0} \mu(B_\gamma) \leq \liminf_{\varepsilon \searrow 0} \mu(B_{m_\varepsilon}) \leq \mu(\mathbb{R}^N_+).$$

Consequently,

$$\inf_{\gamma>0} \mu(B_\gamma) = \mu(\mathbb{R}^N_+). \tag{2.3.6}$$

By (2.3.2) we have successively $\mu(B_{\gamma_1}) = \mu(\gamma_1/\gamma_2 B_{\gamma_2}) = \mu(B_{\gamma_2})$. From (2.3.6) we obtain the conclusion.

□

Proposition 2.3.3. *Assume $\Omega \subset \mathbb{R}^N$ is a smooth domain, which up to a rotation, is supported in $\mathbb{R}^N_+$. Then it holds that*

$$\mu(\Omega) = \mu(\mathbb{R}^N_+) = \frac{N^2}{4}.$$

Proof. Indeed, applying the property (2.3.3) we can assume that $\Omega \subset \mathbb{R}^N_+$, $x_N = 0$ being the tangential hyperplane to Ω. Hence $\mu(\Omega) \geq \mu(\mathbb{R}^N_+)$. Conversely, by the "interior ball property at zero" of Ω, for some $\varepsilon > 0$, there exists a ball B_ε as defined in (2.3.4), such that $B_\varepsilon \subset \Omega$. Therefore we obtain $C(\Omega) \leq \mu(B_\varepsilon) = \mu(\mathbb{R}^N_+)$. This ends the proof.

□

As we said above, the value of the optimal constant $\mu(\Omega)$ depends on the various geometric properties of Ω. In the sequel we assume Ω to be a Lipschitz domain with smooth boundary around the origin. Then $\partial\Omega$ is an $(N-1)$-Riemannian submanifold of $\mathbb{R}^N$ and assume that $\alpha_1, \alpha_2, ..., \alpha_{N-1}$ are the principal curvatures of $\partial\Omega$ at 0. Then, up to a rotation (cf. [18], [74]), the boundary near the origin can be written as

$$x_N = h(x') = \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2) \text{ as } |x'| \to 0, \tag{2.3.7}$$

where $x' = (x_1, ..., x_{N-1}, 0)$. It is well-known that the principal curvatures are the eigenvalues of the $2nd$ fundamental form of the surface $\partial\Omega$ (cf. [70]). If we choose

$$\gamma < \min\{\alpha_i : 1 \leq i \leq N\}, \tag{2.3.8}$$

then $x_N > \gamma|x'|^2$ in Ω for any $(x_N, x') \in \mathbb{R}^N$ very close to origin. Such points belong to the paraboloid P_γ defined by

$$P_\gamma = \{x = (x', x_N) \in \mathbb{R}^N | x_N > \gamma|x'|^2\}. \tag{2.3.9}$$

Due to the considerations above, we distinguish the following four main situations.

C1. $\Omega \subset P_\gamma$, for some $\gamma > 0$ (see Figure 2.1, top left).

C2. $\Omega \subset P_0$, where $P_0 = \mathbb{R}^N_+$ (see Figure 2.1, top right).

C3. $\Omega \subset P_\gamma$, for some $\gamma \geq 0$, in a neighborhood of origin (see Figure 2.1, bottom left).

C4. $\Omega \subset P_\gamma$, for some negative $\gamma < 0$ (see Figure 2.1, bottom right).

Remark 2.3.1. *In cases C1 and C2, Ω lies in $\mathbb{R}^N_+$ (see Figure 2.1, top). From this point of view they may be analyzed together and the results that are true in C2 are also valid in C1. However, we analyze them separately because we present two independent tools to treat each of them.*

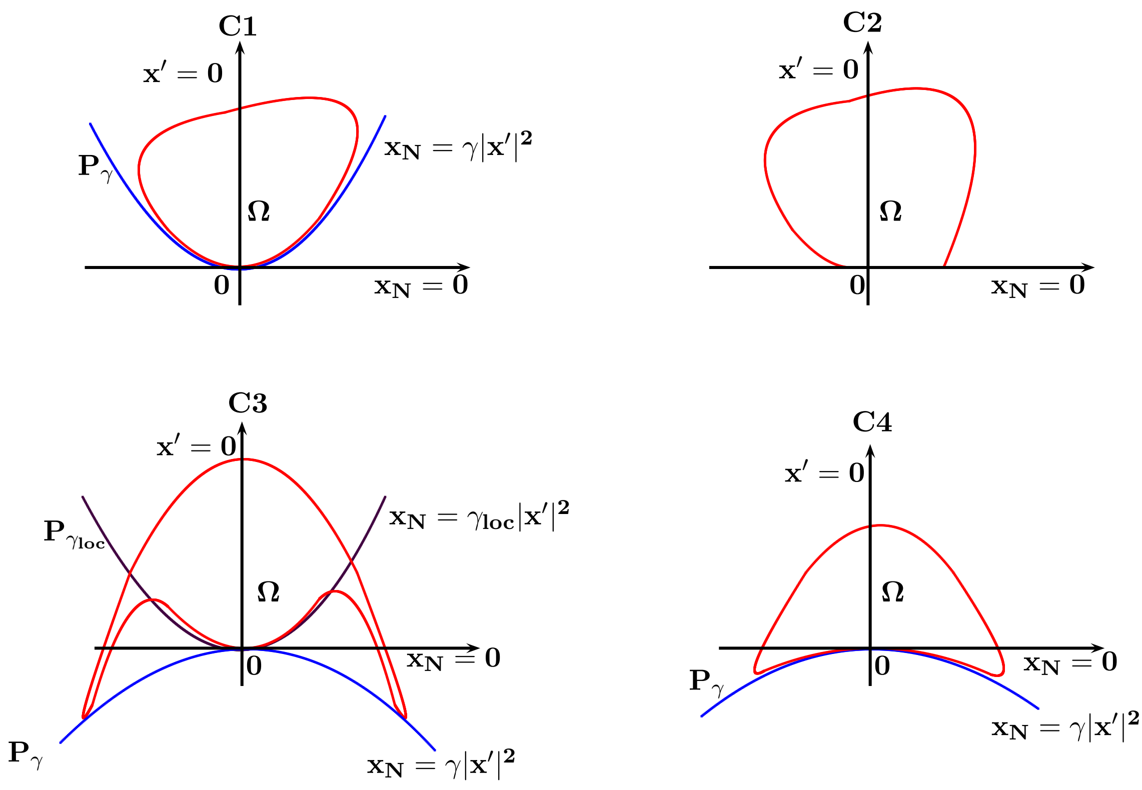

Figure 2.1:

Next we emphasize that the best constant in Hardy inequality in smooth domains depends on the geometry of Ω, and some technical difficulties appear when Ω changes the convexity at origin.

Let $\Omega \subset P_\gamma$ be a domain fulfilling one of the conditions C1-C4 in Figure 2.1, for some constant $\gamma \in \mathbb{R}$. For any constant $C \in \mathbb{R}$, we consider the test function ϕ defined by

$$\phi(x) = (x_N - \gamma|x'|^2)|x|^{-C}. \tag{2.3.10}$$

Remark that such ϕ satisfies the hypothesis of Proposition 2.2.1 in P_γ and in particular in Ω. Moreover, we have

$$\begin{aligned}\nabla\phi = &\sum_{i=1}^{N-1}[-2\gamma x_i|x|^{-C} - C(x_N - \gamma|x'|^2)|x|^{-C-2}x_i]e_i + \\ &+ [|x|^{-C} - Cx_N(x_N - \gamma|x'|^2)|x|^{-C-2}]e_N,\end{aligned}$$

respectively

$$\Delta\phi = (x_N - \gamma|x'|^2)|x|^{-C-2}(C^2 - CN) - 2\gamma[(N-1)|x|^2 - C|x'|^2]|x|^{-C-2}$$

where $\{e_i\}_{i=1,N}$ is canonical basis of $\mathbb{R}^N$. In other words, we get the quotient

$$-\frac{\Delta\phi}{\phi} = \frac{CN - C^2}{|x|^2} + \frac{2\gamma[(N-1)|x|^2 - C|x'|^2]}{(x_n - \gamma|x'|^2)|x|^2}. \tag{2.3.11}$$

Then cf. Proposition 2.2.1 and (2.2.2) we have the inequality

$$\int_\Omega |\nabla v|^2 dx \geq (CN - C^2)\int_\Omega \frac{v^2}{|x|^2}dx + 2\gamma\int_\Omega \frac{((N-1)|x|^2 - C|x'|^2)}{(x_N - \gamma|x'|^2)|x|^2}v^2 dx, \tag{2.3.12}$$

for any $C \in \mathbb{R}$ and all $v \in H_0^1(\Omega)$.

Main results

As a consequence of inequality (2.3.12) we have

Theorem 2.3.1. *Let $N \geq 3$. Assume that Ω satisfies the condition* C1. *Then, there exists a positive constant $C(\Omega, \gamma) > 0$ such that*

$$v \in H_0^1(\Omega), \quad \int_\Omega |\nabla v|^2 dx - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx \geq C(\Omega,\gamma)\int_\Omega \frac{v^2}{x_N - \gamma|x'|^2}dx. \tag{2.3.13}$$

When $N = 2$, the following weaker inequality holds

$$\int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx. \tag{2.3.14}$$

Remark 2.3.2. *From Theorem 2.3.1 we obtain in particular,*

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega |\nabla v|^2 dx - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx \geq D(\Omega,\gamma)\int_\Omega \frac{v^2}{|x|}dx, \tag{2.3.15}$$

for some positive constant $D(\Omega, \gamma) > 0$. In particular this implies that $\mu(\Omega) = N^2/4$.

Proof of Theorem 2.3.1. We choose $C = N/2$ in (2.3.12) since

$$\max_{C\in\mathbb{R}} \{CN - C^2\} = \{CN - C^2\}\Big|_{C=N/2} = \frac{N^2}{4}.$$

Then we obtain

$$\int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx + 2\gamma \int_\Omega \Big((N-1)|x|^2 - \frac{N}{2}|x'|^2\Big)(x_N - \gamma|x'|^2)|x|^{-N-2}u^2 dx \quad (2.3.16)$$

Using that

$$(N-1)|x|^2 - \frac{N}{2}|x'|^2 \geq \frac{N-2}{2}|x|^2,$$

we get

$$\int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx + \gamma(N-2)\int_\Omega \frac{v^2(x)}{x_N - \gamma|x'|^2}dx. \quad (2.3.17)$$

We split the last term in two parts as follows:

$$\int_\Omega \frac{v^2}{x_N - \gamma|x'|^2}dx = \int_{\{x\in\Omega,\ |x'|\leq 1/\gamma\}} \frac{v^2}{x_N - \gamma|x'|^2}dx + \int_{\{x\in\Omega,\ |x'|\geq 1/\gamma\}} \frac{v^2}{x_N - \gamma|x'|^2}dx := I_1 + I_2$$

In the first term, using that $|x'| \leq 1/\gamma$ implies $x_N - \gamma|x'|^2 \leq 2|x|$, we obtain

$$I_1 \geq \frac{1}{2}\int_{\{|x'|\leq 1/\gamma\}} \frac{v^2}{|x|}dx.$$

Using the notation $R_\Omega = \sup_{x\in\overline{\Omega}} |x|$ we have $x_N - \gamma|x'|^2 \leq R_\Omega + \gamma R_\Omega^2$. Thus, for the second term we obtain

$$I_2 \geq \frac{1}{R_\Omega + \gamma R_\Omega^2}\int_{\{|x'|\geq 1/\gamma\}} v^2 dx \geq \frac{1}{\gamma(R_\Omega + \gamma R_\Omega^2)}\int_{\{|x'|\geq 1/\gamma\}} \frac{v^2}{|x|}dx$$

Combining these two lower bounds we get

$$I_1 + I_2 \geq \min\Big\{\frac{1}{2}, \frac{1}{\gamma(R_\Omega + \gamma R_\Omega^2)}\Big\}\int_\Omega \frac{v^2}{|x|}dx,$$

and this, together with (2.3.17) yields (2.3.13). For $N = 2$, (4.3.17) holds easily from (2.3.17). □

Inequality (2.3.12) does not provide sufficient information for $\gamma = 0$. However, using spherical harmonics decomposition, we can extend and improve the result of Theorem 2.3.1 to the case $\gamma \geq 0$ (see Theorem 2.3.2).

Next we state an useful lemma whose proof can be found in [3].

Lemma 2.3.1 (see [3])**.** *Let $L > R > 0$. Then*

$$\int_0^R (w'(r))^2 r dr \geq \frac{1}{4}\int_0^R \frac{w^2}{r^2\log^2(L/r)} r dr, \quad \forall \quad w \in H_0^1(0, R). \quad (2.3.18)$$

Lemma 3.5.6 applies to show Theorem 2.3.2 stated below. This theorem was proved at the same time in [57] using the Emden-Fowler transform.

Theorem 2.3.2. *Let $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ be such that the condition* C2 *is satisfied in Figure 2.1 (top, right). If L is a positive number such that $L > \sup_{x\in\overline{\Omega}} |x|$, then for any $v \in H_0^1(\Omega)$,*

$$\int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx + \frac{1}{4}\int_\Omega \frac{v^2}{|x|^2\log^2(L/|x|)}dx, \tag{2.3.19}$$

Proof of Theorem 2.3.2. Firstly, let us set $R = R_\Omega := \sup_{x\in\overline{\Omega}} |x|$. such that $\Omega \subset B_R^+$ where B_R^+ is the half ball of radius R

$$B_R^+ := \{x \in \mathbb{R}^N,\ |x| \leq R,\ x_N \geq 0\}.$$

We consider also the lower half ball of radius R,

$$B_R^- := \{x \in \mathbb{R}^N,\ |x| \leq R,\ x_N \leq 0\}.$$

The proof follows the idea of decomposition in spherical harmonics (see [77]). By a density argument we can consider $v \in C_0^1(B_R^+)$. Building the odd extension

$$u(x) = u(x_1, x_2, ..., x_N) := \begin{cases} v(x_1, x_2, ..., x_N), & x \in B_R^+, \\ -v(x_1, x_2, ..., -x_N), & x \in B_R^-, \end{cases} \tag{2.3.20}$$

we obtain $u \in C_0^1(B_R)$ and moreover,

$$\int_{B_R^+} |\nabla v|^2 dx = \frac{1}{2}\int_{B_R} |\nabla u|^2 dx, \tag{2.3.21}$$

$$\int_{B_R^+} \frac{v^2}{|x|^2}dx = \frac{1}{2}\int_{B_R} \frac{u^2}{|x|^2}dx. \tag{2.3.22}$$

Next we note that

$$\int_{S^{N-1}} u(r,\sigma)d\sigma = 0, \quad \forall \quad r \in [0,1].$$

Consider the expansion of u in spherical harmonics

$$u(x) = u(r,\sigma) = \sum_{k=0}^{\infty} u_k(r) f_k(\sigma). \tag{2.3.23}$$

Here $(f_k)_{k\geq 0}$ is an ortonormal basis of $L^2(S^{N-1})$ constituted by the eigenvectors of the spherical Laplacian $\Delta_{S^{N-1}}$ with the corresponding eigenvalues $c_k = k(N+k-2)$, $k \geq 0$. It is well-known that f_0 is a constant. Integrating (2.3.23) on S^{N-1} we get

$$u_0(r) = \int_{S^{N-1}} u(r,\sigma) f_0(\sigma) d\sigma = f_0(\sigma)\int_{S^{N-1}} u(r,\sigma)d\sigma = 0,$$

Therefore

$$u(x) = u(r,\sigma) = \sum_{k=1}^{\infty} u_k(r) f_k(\sigma). \tag{2.3.24}$$

and by Plancherel identity we have

$$\int_{B_R} u^2 dx = \sum_{k=1}^{\infty} \int_0^R |u_k(r)|^2 r^{N-1} dr. \tag{2.3.25}$$

Using the representation of the Laplace operator in spherical coordinates

$$\Delta = -\partial_r^2 - \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\Delta_{S^{N-1}},$$

we get

$$\int_{B_R} |\nabla u|^2 dx = \sum_{k=1}^{\infty} \int_0^R \Big[|u_k^{'}|^2 + c_k \frac{u_k^2(r)}{r^2}\Big] r^{N-1} dr. \tag{2.3.26}$$

Let us denote $w_k(r) = u_k(r) r^{\frac{N-2}{2}}$ and $C_N := (N-2)^2/4$. Then by (2.3.26) we have

$$\begin{aligned}\int_{B_R} |\nabla u|^2 dx - \frac{N^2}{4}\int_{B_R} \frac{u^2}{|x|^2} dx &= \sum_{k=1}^{\infty} \int_0^R \Big[|u_k^{'}|^2 - C_N \frac{u^2}{r^2}\Big] r^{N-1} dr + \\ &+ \sum_{k=1}^{\infty} \Big(c_k - (N-1)\Big) \int_0^R \frac{u_k^2}{r^2} r^{N-1} dr \\ &\geq \sum_{k=1}^{\infty} \int_0^R \Big[|u_k^{'}|^2 - C_N \frac{|u_k|^2}{r^2}\Big] r^{N-1} dr \\ &= \sum_{k=1}^{\infty} \int_0^R |w_k'(r)|^2 r dr \end{aligned} \tag{2.3.27}$$

Hence, by (2.3.27) and Lemma 2.3.1 we have

$$\begin{aligned}\int_{B_R} |\nabla u|^2 dx - \frac{N^2}{4}\int_{B_R} \frac{u^2}{|x|^2} dx &\geq \frac{1}{4}\sum_{k=1}^{\infty} \int_0^R \frac{w_k^2(r)}{r^2 \log^2(L/r)} r dr \\ &= \frac{1}{4}\sum_{k=1}^{\infty} \int_0^R \frac{u_k^2(r)}{r^2 \log^2(L/r)} r^{N-1} dr. \end{aligned} \tag{2.3.28}$$

On the other hand

$$\sum_{k=1}^{\infty} \int_0^R \frac{u_k^2(r)}{(r^2 \log^2(L/r)} r^{N-1} dr = \int_{B_R} \frac{u^2}{|x|^2 \log^2(L/|x|)} dx. \tag{2.3.29}$$

By (2.3.28), (2.3.29) and undoing the variables, the proof is completed. □

The techniques presented so far do not fit to domains satisfying the configuration C4 in Figure 2.1. Next we state a more general result due to [57], which is independent on the geometry of Ω near origin (Ω is allowed to be both convex or concave).

Theorem 2.3.3 (see e.g. [57])**.** *Assume $\Omega \subset$ is a smooth domain with the origin on the boundary, there exists a positive constant $r_0 = r_0(\Omega, N) > 0$ such that*

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}. \tag{2.3.30}$$

where $B_{r_0}(0)$ denotes the N-d ball of radius r_0 centered at origin.

The proof of Theorem 2.3.3 in [57] requires differential geometry tools like exponential maps and Fermi coordinates.

In this section we give a direct proof of Theorem 2.3.3 avoiding the technical difficulties in [57]. Our proof is based on some essential ideas in [8].

It is not difficult to notice that in order to show the validity of (3.1.2) it is enough to reduce the proof for two main situations: $\Omega = B_1(e_N)$ and $\Omega = \mathbb{R}^N \setminus \overline{B_1(-e_N)}$. where e_N is the N-th canonical vector in $\mathbb{R}^N$, $e_N = (0, 0, \dots, 0, 1)$. The proof of the first case has been presented above, taking advantage of the convexity of the entire domain. The proof for the second case is more intricate.

Next we focus precisely to show that

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}, \text{ for } \Omega = \mathbb{R}^N \setminus \overline{B_1(-e_N)}, \tag{2.3.31}$$

and some positive constant $r_0 = r_0(\Omega, N) > 0$, fact which yields to the conclusion of Theorem 2.3.3.

Proof of Theorem 2.3.3. In order to conclude the validity of (2.3.31) it suffices to build a function ϕ as in the hypothesis of Proposition 7.0.8 satisfying the pointwise inequality

$$-\frac{\Delta\phi}{\phi} \geq \frac{N^2}{4}, \quad \text{a.e. in } \Omega \cap B_{r_0}(0), \tag{2.3.32}$$

for some $r_0 > 0$ small enough. We do the construction in several steps.

Step 1. We start considering ϕ of the form

$$\phi = f|x|^{\alpha}, \tag{2.3.33}$$

where α and f will be specified later on. For all $i \in \{1, \dots, N\}$ we obtain

$$\phi_{x_i} = f_{x_i}|x|^{\alpha} + \alpha x_i |x|^{\alpha-2} f,$$

$$\phi_{x_i x_i} = f_{x_i x_i}|x|^{\alpha} + f_{x_i}\alpha x_i|x|^{\alpha-2} + \alpha|x|^{\alpha-2} f + \alpha(\alpha-2)x_i^2 n|x|^{\alpha-4} f + \alpha x_i|x|^{\alpha-2} f_{x_i}.$$

Then

$$\begin{aligned}\Delta\phi &= \Delta f|x|^{\alpha} + 2\alpha|x|^{\alpha-2}\nabla f \cdot x + \alpha N|x|^{\alpha-2} f + \alpha(\alpha-2)|x|^{\alpha-2} f \\ &= \Big\{\frac{\Delta f}{f}|x|^2 + 2\alpha\frac{\nabla f \cdot x}{f} + \alpha^2 + \alpha(N-2)\Big\} f|x|^{\alpha-2}\end{aligned} \tag{2.3.34}$$

Therefore,

$$-\frac{\Delta\phi}{\phi} = \Big\{-\frac{\Delta f}{f}|x|^2 - 2\alpha\frac{\nabla f \cdot x}{f} - \alpha^2 - \alpha(N-2)\Big\}\frac{1}{|x|^2} \tag{2.3.35}$$

Next, we call

$$S(f, \alpha) := -\frac{\Delta f}{f}|x|^2 - 2\alpha\frac{\nabla f \cdot x}{f} - \alpha^2 - \alpha(N-2). \tag{2.3.36}$$

The goal reduces to finding a suitable triple (f, α, r_0) such that

$$\sup_{\alpha} S(f,\alpha) \geq \frac{N^2}{4}, \quad \forall |x| < r_0.$$

Note that

$$\sup_{\alpha}\{-\alpha^2-(N-2)\alpha\} = \{-\alpha^2-(N-2)\alpha\}_{\alpha=-N/2} = \frac{N^2}{4} - N.$$

Therefore

$$\sup_{\alpha} S(f,\alpha) = S(f, -\frac{N}{2}) = -\frac{\Delta f}{f}|x|^2 + N\frac{\nabla f\cdot x}{f} + \frac{N^2}{4} - N. \tag{2.3.37}$$

Step 2. We consider f defined by

$$f = d(x)e^{(1-N)d(x)}\Big|\log\frac{1}{|x|}\Big|^{1/2}, \tag{2.3.38}$$

where $d(x) = d(x, \partial\Omega)$ denotes the distance to the boundary.

Next we will show that there exists $r_0 > 0$, $r_0 = r_0(N)$ and a constant $c = c(r_0) > 0$ such that

$$S(f, -\frac{N}{2}) \geq \frac{N^2}{4} + \frac{N-1}{d(x)} + c\Big|\log\frac{1}{|x|}\Big|^{-2}, \quad \forall x \in \Omega\cap B_{r_0}(0). \tag{2.3.39}$$

Proof. Recall that

$$S(f, -\frac{N}{2}) = -\frac{\Delta f}{f}|x|^2 + N\frac{\nabla f\cdot x}{f} + \frac{N^2}{4} - N. \tag{2.3.40}$$

For f as in (2.3.38)we get by direct computations

$$\frac{\nabla f\cdot x}{f} = \frac{\nabla d\cdot x}{d} + (1-N)\nabla d\cdot x - \frac{1}{2}\Big(\log\frac{1}{|x|}\Big)^{-1}. \tag{2.3.41}$$

In order to compute $\Delta f/f$ the following relations hold true

$$\nabla\Big[\Big(\log\frac{1}{|x|}\Big)^{1/2}\Big] = -\frac{1}{2}\frac{x}{|x|^2}\Big(\log\frac{1}{|x|}\Big)^{-1/2}, \tag{2.3.42}$$

$$\Delta\Big[\Big(\log\frac{1}{|x|}\Big)^{1/2}\Big] = -\frac{1}{2}\frac{N-2}{|x|^2}\Big(\log\frac{1}{|x|}\Big)^{-1/2} - \frac{1}{4}\frac{1}{|x|^2}\Big(\log\frac{1}{|x|}\Big)^{-1/2}, \tag{2.3.43}$$

$$\nabla\Big[de^{(1-N)d}\Big] = \nabla de^{(1-N)d} + (1-N)d\nabla de^{(1-N)d} \tag{2.3.44}$$

$$\begin{aligned}\nabla\Big[de^{(1-N)d}\Big] &= \Delta de^{(1-N)d} + 2(1-N)e^{(1-N)d} + d(1-N)\Delta de^{(1-N)d}\\ &\quad + d(1-N)^2e^{(1-N)d}.\end{aligned} \tag{2.3.45}$$

Computing we have

$$\begin{aligned}\Delta f = \Delta[de^{(1-N)d}]\Big(\log\frac{1}{|x|}\Big)^{1/2} &= 2\nabla[de^{(1-N)d}]\cdot\nabla\Big[\Big(\log\frac{1}{|x|}\Big)^{1/2}\Big]\\ + de^{(1-N)d}\Delta\Big[\Big(\log\frac{1}{|x|}\Big)^{1/2}\Big].\end{aligned} \tag{2.3.46}$$

Then taking advantage of formulas (2.3.41)-(2.3.44) we obtain

$$-\frac{\Delta f}{f}|x|^2 = \Big[-\frac{\Delta d}{d} - \frac{2(1-N)}{d} + (N-1)\Delta d - (N-1)^2\Big]|x|^2 \\ + \Big[\frac{\nabla d\cdot x}{d} + (1-N)\nabla d\cdot x\Big]\Big(\log\frac{1}{|x|}\Big)^{-1} + \frac{N-2}{2}\Big(\log\frac{1}{|x|}\Big)^{-1} + \frac{1}{4}\Big(\log\frac{1}{|x|}\Big)^{-2}. \tag{2.3.47}$$

As in [8] we know that $\Delta d(x) = \sum_{i=1}^{N-1}\frac{k_i}{1+k_i d}$, where k_i, $i = 1, N-1$ are the principal curvature of the boundary at $pr(x)$, which is the projection of x on the boundary $\partial\Omega$. In our case, $\partial\Omega = S_1(-e_N)$ and therefore $k_1 = k_2 = \ldots = k_{N-1} = 1$. Thus

$$\Delta d(x) = \frac{N-1}{d(x)}, \quad \forall x\in\Omega.$$

Due to this we have

$$-\frac{2(1-N)}{d} - \frac{\Delta d}{d} \geq \frac{N-1}{d}, \quad \forall x\in\Omega.$$

Besides,

$$\nabla d = -\nu(x),$$

where $\nu(x)$ is the outward normal vector to $\partial\Omega$ at $pr(x)$. Using the formulas above we obtain

$$T := -\frac{\Delta f}{f}|x|^2 + N\frac{\nabla f\cdot x}{f} \geq \Big[\frac{N-1}{d} + \frac{(N-1)^2}{1+d} - (N-1)^2\Big]|x|^2 \\ + \Big[\frac{\nabla d\cdot x}{d} + (1-N)\nabla d\cdot x\Big]\Big(\log\frac{1}{|x|}\Big)^{-1} + \frac{1}{4}\Big(\log\frac{1}{|x|}\Big)^{-2} \\ - \Big(\log\frac{1}{|x|}\Big)^{-1} + N\Big[\frac{\nabla d\cdot x}{d} + (1-N)\nabla d\cdot x\Big] \tag{2.3.48}$$

Taking into account that $\nabla d\cdot x \geq d$ and $|\nabla d| = 1$, for $|x|$ small enough, from above we have

$$T \geq \frac{N-1}{d}|x|^2 + N + \frac{1}{4}\Big(\log\frac{1}{|x|}\Big)^{-2} - C_1|x|^2 - C_2|x|\Big(\log\frac{1}{|x|}\Big)^{-1} - C_3|x| \\ \geq N + \frac{N-1}{d}|x|^2 + \Big(\log\frac{1}{|x|}\Big)^{-2}\Big[\frac{1}{4} - C_2|x|\Big(\log\frac{1}{|x|}\Big) - C_4|x|\Big(\log\frac{1}{|x|}\Big)^2\Big], \tag{2.3.49}$$

for some constants C_1, C_2, C_3, C_4 dependent on N. Finally,

$$\lim_{|x|\to 0}|x|\Big(\log\frac{1}{|x|}\Big) = 0, \quad \lim_{|x|\to 0}|x|\Big(\log\frac{1}{|x|}\Big)^2 = 0,$$

and combining with (2.3.49) we have

$$T := -\frac{\Delta f}{f}|x|^2 + N\frac{\nabla f\cdot x}{f} \geq N + \frac{N-1}{d}|x|^2 + c_0\Big(\log\frac{1}{|x|}\Big)^{-2}, \quad \forall |x|\leq r_0,$$

for some constants $c_0 > 0$, $r_0 > 0$. From this and the expression of $S(f, -\frac{N}{2})$ in (2.3.40) we get the validity of (2.3.39).

Combining (2.3.39) with *Step 1* we have built the function

$$\phi(x) = d(x)e^{(1-N)d(x)}\Big|\log\frac{1}{|x|}\Big|^{1/2}|x|^{-N/2},$$

for which (2.3.32) holds true. Therefore we complete the proof of Theorem 2.3.3.

□

Actually, following the proof of Theorem 2.3.3 we have a more accurate result as follows.

Corolary 2.3.1. *Assume $\Omega = \mathbb{R}^N \setminus \overline{B_1(-e_N)}$. Then, there exists $r_0 > 0$, $r_0 = r_0(N)$ and a constant $c = c(r_0) > 0$ such that*

$$\int_{\Omega\cap B_{r_0}(0)} |\nabla u|^2 dx \geq \int_{\Omega\cap B_{r_0}(0)} \Big(\frac{N^2}{4} + \frac{N-1}{d(x)} + c\Big|\log\frac{1}{|x|}\Big|^{-2}\Big)\frac{u^2}{|x|^2}dx, \quad \forall u \in \Omega\cap B_{r_0}(0).$$

and in particular

$$\int_{\Omega\cap B_{r_0}(0)} |\nabla u|^2 dx \geq \frac{N^2}{4}\int_{\Omega\cap B_{r_0}(0)} \frac{u^2}{|x|^2}dx, \quad \forall u \in \Omega\cap B_{r_0}(0). \tag{2.3.50}$$

Inequality (2.3.50) *finishes the proof of* (2.3.31).

Next we state

Theorem 2.3.4. *Let $N \geq 2$ and Ω be a smooth domain satisfying as in Figure 2.1. Then, there exists a constant $C = C(\Omega, N) < \infty$ such that any $v \in H_0^1(\Omega)$ verifies*

$$C\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx + \frac{1}{4}\int_\Omega \frac{v^2}{|x|^2\log^2(L/|x|)}dx, \tag{2.3.51}$$

where $L = \sup_{x\in\overline{\Omega}}|x|$.

Sketch of the proof of Theorem 2.3.4: We apply a standard cut-off argument so that the function v can be split as $v = v_1 + v_2$ where v_1 lies near the singularity and v_2 is supported away from it. In the neighborhood of $x = 0$, we can apply the improved inequality of Theorem 2.3.2 corresponding to v_1. Outside the origin there are no singularities so that the potential $1/|x|^2$ that appears in the inequality, is bounded by a constant depending only on Ω and the profile of the cut-off function. This fact makes the quantity $\int v_2^2/|x|^2 dx$ to be bounded from above, up to a constant, by $\int v_2^2 dx$. There is also an intermediate zone that we have to deal with, and more precisely where the profile of the cut-off functions has the gradient different by zero. In that part, it suffices to show that the cross term $\int \nabla v_1 \cdot \nabla v_2$ is bounded from below, up to a constant, by $\int v^2 dx$. Gluing these, the proof of (2.3.51) ends. We skip all the computations of the proof but for more details of a cut-off technique see e.g. [111], pp. 111.

In the sequel we consider a domain $\Omega \subset \mathbb{R}^N$ as in Figure 2.1 (bottom, right). The result we obtain is stated as follows.

Proposition 2.3.4. *There exist smooth bounded open sets $\Omega \subset \mathbb{R}^N_+$, $N \geq 2$, satisfying either C3 or C4 and such that*

$$\mu(\Omega) < \frac{N^2}{4}. \tag{2.3.52}$$

Proof of Proposition 2.3.4. From the characterization of the first eigenvalue one can show the strict anti-monotonicity

$$\mathcal{D}_1 \subset\subset \mathcal{D}_2 \Rightarrow \lambda(\mathcal{D}_1) > \lambda(\mathcal{D}_2) \tag{2.3.53}$$

Next we take a cone $\mathcal{C}$ strictly larger than $\mathbb{R}^N_+$. From (2.3.53) and (2.3.54) we obtain

$$\mu(\mathcal{C}) < \mu(\mathbb{R}^N_+) = \frac{N^2}{4}.$$

Therefore, there exists $u \in C_0^\infty(\mathcal{C})$ such that

$$\frac{\int_{\mathcal{C}} |\nabla u|^2 dx}{\int_{\mathcal{C}} u^2/|x|^2 dx} < \frac{N^2}{4}.$$

Denote $K := \overline{\text{supp} u}$. Then $K \subset\subset \mathcal{C}$ and $\text{dist}(K, \partial\mathcal{C}) > 0$. Hence, we can build an open set Ω satisfying either C3 or C4, such that $K \subset \Omega \subset \mathcal{C}$. Hence, $u \in C_0^\infty(\Omega)$ and we get that

$$\mu(\Omega) \leq \frac{\int_{\mathcal{C}} |\nabla u|^2 dx}{\int_{\mathcal{C}} |u|^2/|x|^2 dx} < \frac{N^2}{4}.$$

The proof is completed. □

2.3.2. Conical domains

Firstly, let us consider a Lipschitz connected cone $\mathcal{C} \subset \mathbb{R}^N \setminus \{0\}$ with the vertex at zero. Let $D \subset S^{N-1}$ be the Lipschitz domain such that

$$\mathcal{C} = \{(r, \omega) \mid r \in (0, \infty),\ \omega \in \mathcal{D}\}$$

Let $\mu(\mathcal{C})$ be the best constant in the Hardy inequality. Then (cf. [93])

$$\mu(\mathcal{C}) = \frac{(N-2)^2}{4} + \lambda_1(\mathcal{D}) \tag{2.3.54}$$

where $\lambda_1(\mathcal{D})$ is the Dirichlet principal eigenvalue of the spherical Laplacian $-\Delta_{S^{N-1}}$ on $\mathcal{D}$. In two dimensions it is well-known that (e.g. [28])

$$\lambda_1(\gamma) := \lambda_1(0, \gamma) = (\pi/2\gamma)^2,$$

where $\gamma \in (0, \pi)$ is the amplitude of the conical sector $\mathcal{C}_\gamma = \{(r, \omega) \mid r \in (0, \infty),\ \omega \in (0, 2\gamma)\}$

The aim of this section is mainly devoted to find lower bounds for $\lambda_1(\mathcal{D})$ in higher dimensions $N \geq 3$. In that sense, the definition of a cone in polar coordinates will be used.

The $N-d$ case, $N \geq 3$

For $0 < \gamma < \pi$ we define the N-dimensional cone, with the amplitude γ, denoted by $\mathcal{C}_\gamma$ (Figure 2.2), consisting in all $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ such that, in spherical coordinates (cf. [100], pp. 293),

$$\mathcal{C}_\gamma : \begin{cases} x_1 = r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-2} \cos_{N-1} \\ x_2 = r \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-2} \sin_{N-1} \\ \vdots \\ x_{N-1} = r \sin\theta_1 \sin\theta_2 \\ x_N = r \cos\theta_1 \end{cases} \tag{2.3.55}$$

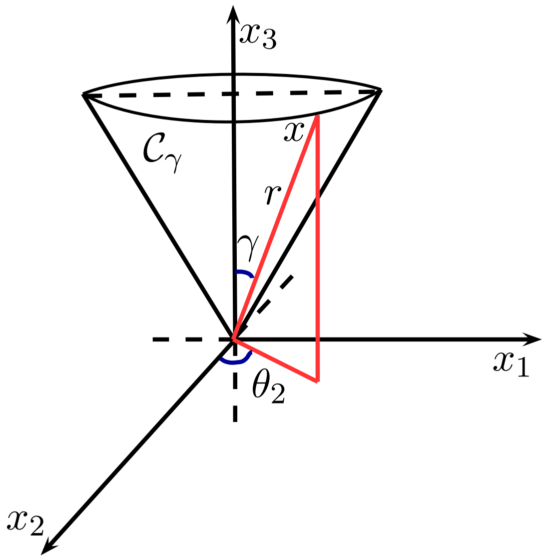

Figure 2.2: The three-dimensional cone with the amplitude γ

with $r > 0$ and

$$\begin{cases} 0 < \theta_1 \leq \gamma, & \\ 0 \leq \theta_i \leq \pi, & \text{for } 2 \leq i \leq N-2, \\ 0 \leq \theta_{N-1} \leq 2\pi. & \end{cases} \tag{2.3.56}$$

For simplicity we denote by $\lambda_1(\gamma) := \lambda_1(D_\gamma)$ the first Dirichlet eigenvalue of the spherical Laplacian on $\mathcal{D}_\gamma := \mathcal{C}_\gamma \cap S^{N-1}$. Then we have

$$\mu(\mathcal{C}_\gamma) = \frac{(N-2)^2}{4} + \lambda_1(\gamma) \tag{2.3.57}$$

In higher dimensions $N \geq 3$, by our knowledge, $\lambda_1(\gamma)$ is well-known only in some few case, in particular the case where $\mathcal{D}_\gamma$ is the semi-sphere S_+^{N-1} mapped in the upper half space $\mathbb{R}_+^N$. More precise, $\lambda_1(S_+^{N-1}) = N-1$. The half space $\mathbb{R}_+^N$ corresponds to the conical sector of amplitude $\gamma = \pi/2$ (see Figure 2.2).

On the other hand (cf. [15]), if $N = 4$ then $\lambda_1(\gamma) = \pi^2/\gamma^2 - 1$. Moreover, to our knowledge, the finest asymptotic behavior of $\lambda_1(\gamma)$ as $\gamma \to 0$ or $\gamma \to \pi$ have been proved in [30].

The best lower and upper bounds for $\lambda_1(\gamma)$ when $\gamma \in (0, \pi/2)$ have been shown in Theorems 4, 5 in [30]. Next we improve the above results showing better lower bounds for $\lambda_1(\gamma)$.

Main results

Theorem 2.3.5. *Assume that $N \geq 3$. If $0 < \gamma \leq \frac{\pi}{2}$ then*

$$\lambda_1(\gamma) \geq \frac{(N-1)\pi^2}{4\gamma^2}.$$

In this section we also give an alternative proof to the asymptotic formula in [30] as γ tends to zero. That is

Theorem 2.3.6 (e.g. [30])**.** *Assume $\gamma \in (0,\pi)$. Then it holds*

$$\lim_{\gamma\to 0}\frac{\lambda_1(\gamma)\gamma^2}{B_1^2}=1,$$

where B_1 is the first positive zero of the Bessel function $J_{\frac{N-3}{2}}$, of fractional order $(N-3)/2$.

Moreover we have

Proposition 2.3.5. *Assume $N\geq 2$. For any fixed $k\in\{1,2,\dots,N\}$ we define the open set*

$$\Omega_k := \{\ x\in\mathbb{R}^N \mid x_1>0, x_2>0,\dots,x_k>0\ \}$$

Then the following Hardy inequality holds true:

$$\int_{\Omega_k}|\nabla u|^2dx \geq \Big(\frac{N-2}{2}+k\Big)^2\int_{\Omega_k}\frac{u^2}{|x|^2}dx \quad \forall u\in H_0^1(\Omega_k). \tag{2.3.58}$$

where the constant $\Big((N-2)/2+k\Big)^2$ is optimal and a minimizing sequence is given by

$$\varepsilon>0,\ u_\varepsilon=\begin{cases} x_1\dots x_k, & |x|\leq 1, x\in\Omega_k,\\ x_1\dots x_k|x|^{-(N-2)/2-k-\varepsilon}, & |x|>1, x\in\Omega_k.\end{cases} \tag{2.3.59}$$

Preliminaries Let us define the N-d line $d_N := \{x\in\mathbb{R}^N \mid x_1=\dots=x_{N-1}\}$. Since $C_0^\infty(\mathcal{C}_\gamma\setminus d_N)$ is dense in $H_0^1(\mathcal{C}_\gamma)$ for all $N\geq 3$, it is enough to restrict the proof to any $u\in C_c^\infty(\mathcal{C}_\gamma\setminus d_N)$. Then, in polar coordinates we have

$$u(r,\theta_1,\theta_2,\dots,\theta_{N-3},\theta_{N-2})\in C^\infty((0,\infty)\times(0,\gamma)\times(0,\pi)\times\dots\times(0,\pi)\times(0,2\pi)), \tag{2.3.60}$$

and moreover

$$u_{|r=\infty}=u_{|\theta_1=0}=u_{|\theta_1=\gamma}=0.$$

The representation of the gradient in polar coordinates is given by

$$\begin{aligned}|\nabla u|^2=|u_r|^2+\frac{u_{\theta_1}^2}{r^2}+\frac{u_{\theta_2}^2}{r^2\sin^2\theta_1}+\frac{u_{\theta_3}^2}{r^2\sin^2\theta_1\sin^2\theta_2}+\\ +\dots+\frac{u_{\theta_{N-1}}^2}{r^2\sin^2\theta_1\sin^2\theta_2\dots\sin^2\theta_{N-2}}.\end{aligned} \tag{2.3.61}$$

In other words,

$$|\nabla u|^2=|u_r|^2+\frac{u_{\theta_1}^2}{r^2}+\text{ positive terms.}$$

The determinant of the Jacobian of the transformation has the form

$$J(r,\theta_1,\theta_2\dots,\theta_{N-2})=r^{N-1}\sin^{N-2}\theta_1\sin^{N-3}\theta_2\dots\sin\theta_{N-2}.$$

To simplify the notations, we define the integral in the variables $\theta_2, \dots, \theta_{N-2}$ as

$$\int := \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \sin^{N-3}\theta_2 \dots \sin\theta_{N-2} \mathrm{d}\theta_2 \dots \mathrm{d}\theta_{N-2} \mathrm{d}\theta_{N-1}.$$

For a fixed $\sigma \in S^{N-1} \cap \mathcal{C}_\gamma$ the radial function $r \mapsto u(\cdot, \sigma)$ satisfies the well-known Hardy inequality

$$\int_0^\infty u_r^2 r^{N-1} dr \geq \Big(\frac{N-2}{2}\Big)^2 \int_0^\infty \frac{u^2}{r^2} r^{N-1} dr,$$

and therefore

$$\int_0^\infty \int_0^\gamma \int u_r^2 r^{N-1} \sin^{N-2}\theta_1 \mathrm{d}\theta_1 dr \geq \Big(\frac{N-2}{2}\Big)^2 \int_0^\infty \int_0^\gamma \int \frac{u^2}{r^2} r^{N-1} \sin^{N-2}\theta_1 \mathrm{d}\theta_1 dr. \quad (2.3.62)$$

From above, due to the lack of the boundary conditions in the variables $\theta_2, \dots, \theta_{N-2}$ we get that $\lambda_1(\gamma)$ is the optimal constant in the weighted inequality

$$\int_0^\infty \int_0^\gamma \int \frac{u_{\theta_1}^2}{r^2} r^{N-1} \sin^{N-2}\theta_1 \mathrm{d}\theta_1 dr \geq \lambda_1(\gamma) \int_0^\infty \int_0^\gamma \int \frac{u^2}{r^2} r^{N-1} \sin^{N-2}\theta_1 \mathrm{d}\theta_1 dr. \quad (2.3.63)$$

More precisely, $\lambda_1(\gamma)$ may be characterized by

$$\lambda_1(\gamma) = \inf_{\{u \in H, u \neq 0\}} \frac{\int_0^\gamma u_{\theta_1}^2 \sin^{N-2}\theta_1 \mathrm{d}\theta_1}{\int_0^\gamma u^2 \sin^{N-2}\theta_1 \mathrm{d}\theta_1}, \quad (2.3.64)$$

where H is the completion of $C_0^1(0, \gamma)$ in the norm

$$||u||_H^2 = \int_0^\gamma u_{\theta_1}^2 \sin^{N-2}\theta_1 \mathrm{d}\theta_1. \quad (2.3.65)$$

Indeed, Let us consider $a_n = a_n(\theta_1)$ a minimizing sequence in (2.3.64). Then the sequence

$$u_n := u_1(r, \theta_2, \dots, \theta_{N-2}) a_n(\theta_1)$$

minimizes $\lambda_1(\gamma)$, provided u_1 is smooth and $u_1 | r = \infty = 0$.

Proofs of Theorems 2.3.5 and 2.3.5

Proof of Theorem 2.3.5. Without losing the generality, we are going to consider u as in (2.3.60). Next we propose the change of variables

$$v(r, \theta_1, \theta_2, \dots, \theta_{N-2}) := u(r, \theta_1, \theta_2, \dots, \theta_{N-2}) / \cos(\frac{\pi}{2\gamma}\theta_1).$$

For simplicity, we write $u(r)$ or $u(\theta_1)$ when referring to the radial variable respectively at the angular part θ_1. Thus, integrating by parts we get the following identity:

$$\begin{aligned} \int_0^\gamma u_{\theta_1}^2(\theta_1) \sin^{N-2}\theta_1 \mathrm{d}\theta_1 &= \int_0^\gamma v_{\theta_1}^2(\theta_1) \cos^2(\frac{\pi}{2\gamma}\theta_1) \sin^{N-2}\theta_1 \mathrm{d}\theta_1 + \frac{\pi^2}{4\gamma^2} \int_0^\gamma u^2(\theta_1) \sin^{N-2}\theta_1 \mathrm{d}\theta_1 \\ &+ (N-2)\frac{\pi}{2\gamma} \int_0^\gamma v^2(\theta_1) \sin(\frac{\pi}{2\gamma}\theta_1) \cos(\frac{\pi}{2\gamma}\theta_1) \cos\theta_1 \sin^{N-3}\theta_1 \mathrm{d}\theta_1. \end{aligned} \quad (2.3.66)$$

Using the identity (2.3.66) and the characterization of $\lambda_1(\gamma)$ stated in (2.3.64), it is enough to show that

$$\cos(\frac{\pi}{2\gamma}\theta_1)\sin(\frac{\pi}{2\gamma}\theta_1)\cos\theta_1 \geq \frac{\pi}{2\gamma}\cos^2(\frac{\pi}{2\gamma}\theta_1)\sin\theta_1, \quad \forall \quad \theta_1 \in [0,\gamma].$$

Obviously, this is true for $\theta_1 = \{0,\gamma\}$. Dividing by $\frac{\pi}{2\gamma}\theta_1\cos^2(\frac{\pi}{2\gamma}\theta_1)\cos\theta_1$ it remains to prove that

$$\frac{\tan(\frac{\pi}{2\gamma}\theta_1)}{\frac{\pi}{2\gamma}\theta_1} \geq \frac{\tan\theta_1}{\theta_1}, \quad \forall\theta_1 \in (0,\gamma).$$

Because $\pi/2\gamma > 1$, the last inequality is true due to the increasing monotonicity of the function $\theta_1 \mapsto \tan\theta_1/\theta_1$ in the interval $(0,\pi/2)$. This ends the proof. □

Proof of Theorem 2.3.6. The main goal is to finding upper and lower bounds for the value $\lambda_1(\gamma)$ in (2.3.64). Due to standard inequalities

$$\frac{\sin\gamma}{\gamma}t \leq \sin t \leq t, \quad \forall t \in (0,\gamma), \forall\gamma \in (0,\pi), \tag{2.3.67}$$

it suffices to determine the value

$$\lambda_\star^1(\gamma) := \inf_{\{u\in H, u\neq 0\}} \frac{\int_0^\gamma u_t^2 t^{N-2}dt}{\int_0^\gamma u^2 t^{N-2}dt}, \tag{2.3.68}$$

By Proposition 2.3.6, it holds that $\lambda_\star^1(\gamma) > 0$. Due to the compact embedding (see Proposition 2.3.7)

$$H \hookrightarrow L^2((0,\gamma); t^{N-2}dt),$$

$\lambda_\star^1(\gamma)$ is attained by a non-trivial function ϕ_1. Then one can prove that $\lambda_1^\star(\gamma)$ satisfies the variational problem: there exists $\phi^1 \in H$ such that

$$\int_0^\gamma \phi_t^1 v_t t^{N-2}dt = \lambda_1^\star(\gamma)\int_0^\gamma \phi^1 v t^{N-2}dt, \quad \forall v \in H. \tag{2.3.69}$$

Next we note that any $u \in H$ exhibits a hidden weak Neumann boundary condition at the origin $t = 0$:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon u_t t^{N-2}dt = 0. \tag{2.3.70}$$

Indeed, we have

$$\Big|\frac{1}{\varepsilon}\int_0^\varepsilon u_t t^{N-2}dt\Big| \leq \frac{1}{\varepsilon}\Big(\int_0^\varepsilon (u_t)^2 t^{N-2}dt\Big)^{1/2}\Big(\int_0^\varepsilon t^{N-2}dt\Big)^{1/2} = \varepsilon^{(N-3)/2}\Big(\int_0^\varepsilon u_t t^{N-2}dt\Big)^{1/2} \tag{2.3.71}$$

which converges to 0 when ε tends to 0. This allows to make integrations by parts and rewrite (2.3.69) as

$$\int_0^\gamma -(\phi_t^1 t^{N-2})_t v dt = \lambda_1^\star(\gamma)\int_0^\gamma \phi^1 v t^{N-2}dt, \quad \forall v \in H. \tag{2.3.72}$$

Therefore, $\lambda_\star^1(\gamma)$ is the first eigenvalue of the degenerate Sturm-Liouville problem

$$\begin{cases} -(u_t t^{N-2})_t = \lambda u t^{N-2}, & t \in (0,\gamma), \\ \lim_{t\to 0} u_t t^{N-2} = 0, \quad u(\gamma) = 0, & \end{cases} \tag{2.3.73}$$

with the corresponding eigenvector ϕ_1.

In the sequel we determine explicitly the value of $\lambda_\star^1(\gamma)$.

With the change of variables $v = ut^{N-2}$, the problem (2.3.73) reduces to the following Bessel equation with boundary constraint

$$\begin{cases} v_{tt} + (2-N)\frac{v_t}{t} + \Big(\lambda + \frac{N-2}{t^2}\Big)v = 0, \quad t \in (0,\gamma), \\ v(\gamma) = 0, \end{cases} \tag{2.3.74}$$

Bessel functions. If n is positive integer then, the first Bessel function J_n of order n has the expression

$$J_n(x) = \frac{x^n}{2^n n!}\Big(1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)\cdot(2n+4)} - \dots\Big) \tag{2.3.75}$$

and J_n behaves like x^n when $x>0$ is small. If n is a negative integer, by definition yields

$$J_{-n}(x) = (-1)^n J_n(x).$$

If n is not an integer then

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\Big(1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)\cdot(2n+4)} - \dots\Big),$$

where Γ denotes the Gamma-function. When n is an integer it is necessary to recall the so-called Weber's function, i.e.

$$Y_n(x) = J_n(x)\int \frac{dx}{xJ_n^2(x)},$$

which behaves like $1/x^n$ when $x>0$ is small. Next, we consider the Bessel equation

$$y_{tt} - (2\alpha-1)\frac{y_t}{t} + \Big(\beta^2\tau^2 t^{2\tau-2} + \frac{\alpha^2 - n^2\tau^2}{t^2}\Big)y = 0, \tag{2.3.76}$$

Due to [20], pp. 117, the general solution of (2.3.76) is given by

$$y = t^\alpha\{AJ_n(\beta t^\tau) + BY_n(\beta t^\tau)\},$$

$$y = t^\alpha\{AJ_n(\beta t^\tau) + BJ_{-n}(\beta t^\tau)\},$$

where A, B are constants, according as n is non-negative integer or not.

Once $\lambda \neq 0$ is an eigenvalue for (2.3.73) then λ is also an eigenvalue in (2.3.74). The general solution of (2.3.74) is a particular case of (2.3.76) for $\alpha = (N-1)/2$, $\tau = 1$, $\beta = \sqrt{\lambda}$, $n = (N-3)/2$, i.e.,

$$v(t) = t^{\frac{N-1}{2}}\{AJ_{\frac{N-3}{2}}(\sqrt{\lambda}t) + BY_{\frac{N-3}{2}}(\sqrt{\lambda}t)\},$$

or

$$v(t) = t^{\frac{N-1}{2}}\{AJ_{\frac{N-3}{2}}(\sqrt{\lambda}t) + BJ_{-\frac{N-3}{2}}(\sqrt{\lambda}t)\}.$$

We show that it must be $B = 0$, in which case it simplifies the expression of v i.e.

$$v(t) = At^{\frac{N-1}{2}} J_{\frac{N-3}{2}}(\sqrt{\lambda}t). \tag{2.3.77}$$

Indeed, if $N = 3$ then, reconsidering the constants, it is trivial that v is as in (2.3.77). Assume $N \geq 4$ and $B \neq 0$. Using the behavior of J_n and Y_n at zero we get that

$$v(t) \sim AC_1 t^{N-1} + BC_2 t,$$

where $C_1 = C_1(\lambda)$, $C_2 = C_2(\lambda)$ are non-trivial constants depending on λ. Consequently, up to a constant,

$$u(t) \sim 1 + Ct^{3-N}$$

with $C \neq 0$. Since $N \geq 4$, this last formula yields to

$$\int_0^\gamma u^2(t)dt = \infty,$$

which contradicts the fact that $u \in H$. Hence, the assumption is false and $B = 0$ for any $N \geq 4$.

Imposing the condition $v(\gamma) = 0$ in the simplified expresion (2.3.77), we obtain $\sqrt{\lambda} = B_n$, where $\{B_n\}_n$ are the positive zero's of the Bessel function $J_{\frac{N-3}{2}}$. In particular we obtain

$$\lambda_\star^1(\gamma) = B_1^2/\gamma^2.$$

Using this, the relations (2.3.64), (2.3.68) and the inequality (2.3.67) we obtain the conclusion of Theorem 2.3.6. □

Proof of Proposition 2.3.5. According to Proposition 2.3.32 it is enough to find proper ϕ such that ϕ satisfying

$$-\frac{\Delta\phi}{\phi} \geq \Big(\frac{N-2}{2} + k\Big)^2 \frac{1}{|x|^2}, \text{ a.e. in } \Omega_k. \tag{2.3.78}$$

Indeed, $\phi = x_1 \dots x_k |x|^{-(N-2)/2-k}$ satisfies (2.3.78) and this yields the proof of Theorem 2.3.5. The optimality remains to be checked.

□

Proofs of useful propositions

Proposition 2.3.6. *For any $v \in H$ we have*

$$\int_0^\gamma v_t^2 t^{N-2} dt \geq \frac{1}{\gamma}\Big(\frac{N-2}{2}\Big)^2 \int_0^\gamma v^2 t^{N-3} dt. \tag{2.3.79}$$

Proof. Of course, we have

$$\int_0^\gamma v_t^2 t^{N-2} dt \geq \frac{1}{\gamma}\int_0^\gamma v_t^2 t^{N-1} dt,$$

and applying the N-d Hardy inequality in the form

$$\int_0^\gamma v_t^2 t^{N-1} dt \geq \Big(\frac{N-2}{2}\Big)^2 \int_0^\gamma v^2 t^{N-3},$$

we complete the proof.

□

Proposition 2.3.7. *The embedding*

$$H \hookrightarrow L^2((0,\gamma);t^{N-2}dt)$$

is compact.

Proof. The key point is played by Proposition 2.3.6.

Let us consider a sequence $(u_n)_n \rightharpoonup 0$ in H and we will show that $u_n \to 0$ in $L^2((0,\gamma);t^{N-2}dt)$.

Firstly we deduce $\{u_n\}_n$ is bounded in $H_0^1((0,\gamma);t^{N-2}dt)$, let's say by a constant C. Accordingly to Proposition 2.3.6 we have as well

$$\int_0^\gamma u_n^2 t^{N-3}dt \le C, \quad \forall n \in \mathbb{N}. \tag{2.3.80}$$

Let $\varepsilon > 0$ be fixed. Then we split the L^2-norm as

$$||u_n||^2_{L^2((0,\gamma);t^{N-2}dt)} = \int_0^{2\varepsilon} |u_n|^2 t^{N-2}dt + \int_{2\varepsilon}^\gamma |u_n|^2 t^{N-2}dt := I^1_{\varepsilon,n} + I^2_{\varepsilon,n}. \tag{2.3.81}$$

Let us also consider the partition of unity of u_n,

$$u_n = u_n\varphi + (1-\varphi)u_n := w_{1,n} + w_{2,n},$$

where φ is a regular function such that

$$\varphi(t) = \begin{cases} 1, & t \le \varepsilon, \\ 0, & t \ge 2\varepsilon, \end{cases} \tag{2.3.82}$$

Obviously, $\mathrm{supp}(w_{1,n}) \subset (0, 2\varepsilon)$, $\mathrm{supp}(w_{2,n}) \subset (\varepsilon,\gamma)$. Firstly, from (2.3.80) we have

$$I^1_{\varepsilon,n} \le 2\varepsilon \int_0^{2\varepsilon} |u_n|^2 t^{N-3}dt \le 2\varepsilon \int_0^\gamma |u_n|^2 t^{N-3}dt \tag{2.3.83}$$

Secondly, let us notice that

$$w_{2,n} \rightharpoonup 0 \text{ in } H_0^1((0,\gamma);t^{N-2}dt). \tag{2.3.84}$$

For this, it suffices to prove $(\psi, w_{2,n})_{H_0^1((0,\gamma);t^{N-2}dt)} \to 0$ for all $\psi \in C_c^\infty$. We evaluate,

$$(\psi, w_{2,n})_{H_0^1((0,\gamma);t^{N-2}dt)} = \int_0^\gamma \psi_t((1-\varphi)u_n)_t t^{N-2}dt = -\int_0^\gamma (\psi_t t^{N-2})_t (1-\phi)u_n dt$$

which converges to 0 when $n \to \infty$. This happens because weak convergence in H involves weak convergence in $L^2((0,\gamma);t^{N-3}dt)$ (by Proposition 2.3.6). Now we observe that the support of $w_{2,n}$ lies far from zero and therefore the norm of $w_{2,n}$ in H is equivalent to the norm of $w_{2,n}$ in $H_0^1(\varepsilon,\gamma)$. But $H_0^1(\varepsilon,\gamma)$ is compact embedded in $L^2(\varepsilon,\gamma)$, and in particular in $L^2((0,\gamma);t^{N-3}dt)$. We obtain that $w_{2,n} \to 0$ in $L^2((0,\gamma);t^{N-3}dt)$. Hence, we can choose n large enough such that $I^2_{\varepsilon,n} < \varepsilon$. From here and (2.3.83), we conclude that u_n converges strongly to 0 in $L^2((0,\gamma);t^{N-2}dt)$.

□

Proposition 2.3.8. *Let us consider $a < 1$. Then, the application*

$$(0, \frac{\pi}{2}) \ni t \to \frac{\tan at}{\tan t} \in (0, \infty)$$

is decreasing.

Proof. Indeed, if we consider $f(t) = \frac{\tan at}{\tan t}$ we obtain

$$f'(t) = \frac{at}{\cos^2 at \sin^2 t}\Big(\frac{\sin 2t}{2t} - \frac{\sin 2at}{2at}\Big).$$

It follows that $f' < 0$ due to the decreasing behavior of the function $x \mapsto \frac{\sin x}{x}$ on $(0, \pi)$.

With this, we complete the proof.

□

2.4. Multipolar Hardy inequalities

This section is concerned with a class of Schrödinger operators of the form $-\Delta + V(x)$ with multipolar Hardy-type singular potentials like $V \sim \sum_i \alpha_i/|x - x_i|^2$, $\alpha_i \in \mathbb{R}$, $x_i \in \mathbb{R}^N$, $N \geq 3$.

Throughout this section we study the qualitative properties of Schrödinger operators with inverse square potentials V, improving some results already known in the literature. In the case of a multi-singular potential $V(x) = \sum_{i=1}^n \alpha_i/|x - x_i|^2$ with $\alpha_i \in \mathbb{R}$, where $x_i \in \mathbb{R}^N$ are the singular poles assumed to be fixed, the study of positivity of the quadratic functional

$$\mathcal{D}[u] = S_{\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_n}[u] := \int_\Omega |\nabla u|^2 dx - \sum_{i=1}^n \alpha_i \int_\Omega \frac{u^2}{|x - x_i|^2} dx \tag{2.4.1}$$

is much more intricate since the interaction among the poles and their configuration matters.

Among other results, in [64] it was proved that when $\Omega = \mathbb{R}^N$, $\mathcal{D}$ is positive if and only if $\sum_{i=1}^n \alpha_i^+ \leq (N-2)^2/4$ for any configuration of the poles $x_1, \ldots, x_n$, where $\alpha^+ = \max\{\alpha, 0\}$. Conversely, if $\sum_{i=1}^n \alpha_i^+ > (N-2)^2/4$, there exist configurations $x_1, \ldots, x_n$ for which $\mathcal{D}$ is negative. These results have been improved later on by Bosi, Dolbeault, Esteban [19] when deriving lower bounds of the spectrum of the operator $-\Delta - \mu \sum_{i=1}^n 1/|x - x_i|^2$, $\mu \in (0, (N-2)^2/4]$, with $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$. Rougly speaking, they showed that for any $\mu \in (0, (N-2)^2/4]$ and any configuration $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$, $n \geq 2$, there is a nonnegative constant $K_n < \pi^2$ such that

$$u \in C_0^\infty(\mathbb{R}^N), \quad \frac{K_n + (n+1)\mu}{d^2} \int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{u^2}{|x - x_i|^2} dx \geq 0, \tag{2.4.2}$$

where d denotes $d := \min_{i \neq j} |x_i - x_j|/2$. The original proof of (2.4.2) in [19] employs a partition of unity technique, the so-called "IMS" (for Ismagilov, Morgan-Simon, Sigal, see [90], [99]), localizing the singular Schrödinger operator. Inequality (2.4.2) emphasizes that we can reach the critical singular mass $(N-2)^2/(4|x-x_i|^2)$ at any singular pole x_i to the prize of adding a lower order term in L^2-norm.

To simplify the notations, here and throughout the section when writing $\int \cdot dx$ we denote the integral over $\mathbb{R}^N$. Besides, using the so-called "expansion of the square" method, the authors in [19] proved the following inequality without lower order terms

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx + \frac{(N-2)^2}{4n^2} \sum_{1\leq i<j\leq n} \int \frac{|x_i-x_j|^2}{|x-x_i|^2|x-x_j|^2} u^2 dx, \tag{2.4.3}$$

for any $u \in H^1(\mathbb{R}^N)$ and any set of poles $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$, $n \geq 2$. Let us denote the singular potentials in (2.4.3) by

$$V_i(x) := \frac{1}{|x-x_i|^2}, \quad V_{ij}(x) := \frac{|x_i-x_j|^2}{|x-x_i|^2|x-x_j|^2}, \quad \forall i,j \in \{1,\ldots,n\}, \quad i \neq j. \tag{2.4.4}$$

Observe that both potentials in (5.3.40) have a quadratic singularity at each pole x_i, i.e.

$$\lim_{x\to x_i} V_i(x)|x-x_i|^2 = 1, \quad \lim_{x\to x_i} V_{ij}(x)|x-x_i|^2 = 1, \quad \forall i,j \in \{1,\ldots,n\}, \quad i \neq j. \tag{2.4.5}$$

Moreover, due to the symmetry we notice that

$$\sum_{1\leq i<j\leq n} V_{ij}(x) = \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{2} V_{ij}(x) = \sum_{i=1}^{n} \frac{\alpha_i(x)}{|x-x_i|^2}, \tag{2.4.6}$$

where

$$\alpha_i(x) = \frac{1}{2} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|x_i-x_j|^2}{|x-x_j|^2} \sim \frac{n-1}{2}, \quad \text{as } x \to x_i, \quad \forall i \in \{1,\ldots,n\}.$$

Therefore, we remark that the total mass arising at a singular pole x_i in (2.4.3) is proportional to

$$\frac{(N-2)^2}{4n} V_i(x) + \frac{(N-2)^2}{4n^2} \sum_{1\leq i<j\leq n}^{n} V_{ij}(x) \sim \frac{(N-2)^2}{4} \frac{3n-1}{2n^2} \frac{1}{|x-x_i|^2}, \tag{2.4.7}$$

as $x \to x_i$, $i \in \{1,\ldots,n\}$. Note however that the multiplicative factor in each singularity in (5.3.18) is smaller than the optimal one that (2.4.2) yields for $\mu = (N-2)^2/4$. This is so because in (2.4.3) no other corrected terms are added.

We also mention the articles [19], [63], [64], [2] and the references therein for other inequalities with multipolar singularities.

In this section we develop new optimal Hardy-type inequalities with multipolar potentials.

In Section 2.4.1 we complement and improve some results in [19] related to inequality (2.4.3). Our proofs use convenient transformations involving the product of the fundamental solutions E_i of the Laplacian at the poles x_i, $i \in \{1,\ldots,n\}$. In Theorem 2.4.1, of Section 2.4.1 we give an optimal inequality for the operator $A_\lambda = -\Delta - \lambda \sum_{1\leq i<j\leq n} V_{ij}(x)$, $\lambda > 0$, showing a better singular behavior of the potential at each pole x_i than pointed out in (2.4.3)-(5.3.18). This allows to show the existence of bounded domains in which, for the bipolar Hardy inequality, the L^2-reminder term in (2.4.2) can be removed. For this to be done, the best situation seems to be the case in which the singularities are localized on the boundary of the domain, as emphasized in Subsection (2.4.2), Proposition 2.4.1.

2.4.1. Main results

Assume $N \geq 3$ and consider n poles $x_1, \dots x_n \in \mathbb{R}^N$, $n \geq 2$, such that $x_i \neq x_j$ for any $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$. In the sequel we improve the result (2.4.3) by Bosi et al. [19] as follows. The main result of this section is as follows.

Theorem 2.4.1. *It holds that*

$$\int |\nabla u|^2 \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int \Big| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \Big|^2 u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.4.8}$$

or equivalently

$$\int |\nabla u|^2 \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{2.4.9}$$

Moreover, the constant $(N-2)^2/n^2$ is optimal.

In the sequel, we prove Theorem 2.4.1 applying identity (7.0.8) before.

Proof of Theorem 2.4.1.

By density arguments it is sufficient to prove (2.4.8) for any function $u \in C_0^1(\mathbb{R}^N \setminus \{x_1, \dots, x_n\})$. Then, according to (7.0.8)-(2.2.3), it is enough to find ϕ satisfying

$$-\frac{\Delta \phi}{\phi} \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \Big| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \Big|^2, \quad \forall x \in \mathbb{R}^N \setminus \{x_1, \dots, x_n\}.$$

Let us choose

$$\phi = E^{1/n} = \prod_{i=1}^n E_i^{1/n}, \tag{2.4.10}$$

where E_i is the fundamental solution of the Laplacian at the singular pole x_i, $i \in \{1, \dots, n\}$, i.e. $E_i = 1/(\omega_N(N-2))|x - x_i|^{2-N}$ where ω_N denotes the $(N-1)$-Hausdorff measure of the unit sphere S^{N-1} in $\mathbb{R}^N$. Note that ϕ chosen in (2.4.10) verifies the integrability conditions to validate the identity (7.0.8). On the other hand, we have

$$\nabla E = \Big(\sum_{i=1}^n \frac{\nabla E_i}{E_i} \Big) E, \quad \forall x \in \mathbb{R}^N \setminus \{x_1, \dots, x_n\}. \tag{2.4.11}$$

Due to the fact that $-\Delta E_i = \delta_{x_i}$ for all $i \in \{1, \dots, n\}$ we obtain

$$\begin{aligned} \Delta E &= \Big(\sum_{i=1} \frac{\Delta E_i}{E_i} + 2 \sum_{1 \leq i < j \leq n} \frac{\nabla E_i \cdot \nabla E_j}{E_i E_j} \Big) E \\ &= 2 \Big(\sum_{1 \leq i < j \leq n} \frac{\nabla E_i \cdot \nabla E_j}{E_i E_j} \Big) E, \quad \forall x \in \mathbb{R}^N \setminus \{x_1, \dots, x_n\}. \end{aligned} \tag{2.4.12}$$

Combining (2.4.10), (2.4.11) and (2.4.12) we notice that ϕ satisfies precisely the equation

$$-\Delta\phi - \frac{(N-2)^2}{n^2}\sum_{1\le i<j\le n}\left|\frac{x-x_i}{|x-x_i|^2}-\frac{x-x_j}{|x-x_j|^2}\right|^2\phi = 0, \quad \forall x\in\mathbb{R}^N\setminus\{x_1,\ldots,x_n\}. \quad (2.4.13)$$

Then, identity (7.0.8) becomes

$$\int\Big[|\nabla u|^2 - \frac{(N-2)^2}{n^2}\sum_{1\le i<j\le n}\left|\frac{x-x_i}{|x-x_i|^2}-\frac{x-x_j}{|x-x_j|^2}\right|^2 u^2\Big]dx$$
$$= \int\left|\nabla u - \frac{\nabla(E^{1/n})}{E^{1/n}}u\right|^2 dx = \int |\nabla(uE^{-1/n})|^2 E^{2/n}dx \ge 0. \quad (2.4.14)$$

This concludes the proof of (2.4.8).

Optimality of the constant.

Next we complete the proof of Theorem 2.4.1 by showing the optimality of the constant $(N-2)^2/n^2$ in (2.4.8).

According to (2.4.14), we actually showed that for all $u\in H^1(\mathbb{R}^N)$ we have

$$\int|\nabla u|^2dx - \frac{(N-2)^2}{n^2}\sum_{1\le i<j\le n}\int\left|\frac{x-x_i}{|x-x_i|^2}-\frac{x-x_j}{|x-x_j|^2}\right|^2u^2dx = \int|\nabla(uE^{-1/n})|^2E^{2/n}dx. \quad (2.4.15)$$

Here $B_r(x)\subset\mathbb{R}^N$, for some fixed $r>0$ and $x\in\mathbb{R}^N$, denotes the ball of radius r centered at x.

For $\varepsilon>0$ aimed to be small ($\varepsilon<\min\{1,d/2\}$), we consider the cut-off functions $\theta_\varepsilon\in C_0(\mathbb{R}^N)$ defined by

$$\theta_\varepsilon(x) = \begin{cases} 0, & |x-x_i|\le\varepsilon^2, \quad \forall i\in\{1,\ldots,n\},\\ \frac{\log|x-x_i|/\varepsilon^2}{\log 1/\varepsilon}, & \varepsilon^2\le|x-x_i|\le\varepsilon, \quad \forall i\in\{1,\ldots,n\},\\ 1, & x\in B_{1/\varepsilon}(0)\setminus\cup_{i=1}^n B_\varepsilon(x_i),\\ \varepsilon\big(\frac{2}{\varepsilon}-|x|\big), & 1/\varepsilon\le|x|\le 2/\varepsilon,\\ 0, & |x|\ge 2/\varepsilon.\end{cases} \quad (2.4.16)$$

Then we consider the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ defined by

$$u_\varepsilon := E^{1/n}\theta_\varepsilon, \quad \varepsilon>0,$$

which belongs to $C_0(\mathbb{R}^N)$ ($\subset H^1(\mathbb{R}^N)$) since θ_ε belongs to $C_0(\mathbb{R}^N)$ and is supported far from the poles x_i.

In the sequel we show that $\{u_\varepsilon\}_{\varepsilon>0}$ is an approximating sequence for $(N-2)^2/n^2$, that is

$$\lim_{\varepsilon\searrow 0}\frac{\int|\nabla u_\varepsilon|^2dx}{\sum_{1\le i<j\le n}\int\left|\frac{x-x_i}{|x-x_i|^2}-\frac{x-x_j}{|x-x_j|^2}\right|^2u_\varepsilon^2dx} = \frac{(N-2)^2}{n^2}. \quad (2.4.17)$$

Firstly, we can easily notice that there exists a constant $C > 0$ depending on d (uniformly in ε) such that

$$\sum_{1\leq i<j\leq n} \int \Big|\frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2}\Big|^2 u_\varepsilon^2 dx > C, \quad \forall \varepsilon > 0. \tag{2.4.18}$$

On the other hand, taking into account where $\nabla\theta_\varepsilon$ is supported, we split in two parts

$$\begin{aligned}\int |\nabla(u_\varepsilon E^{-1/n})|^2 E^{2/n} dx &= \sum_{i=1}^n \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} |\nabla\theta_\varepsilon|^2 E^{2/n} dx + \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} |\nabla\theta_\varepsilon|^2 E^{2/n} dx \\ &:= I_1 + I_2. \end{aligned}\tag{2.4.19}$$

Next we obtain

$$I_1 = \frac{1}{\omega_N^2 (N-2)^2} \sum_{i=1}^n \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} \frac{1}{\log^2(1/\varepsilon)} \frac{1}{|x-x_i|^2} \prod_{j=1}^n |x-x_j|^{2(2-N)/n} dx. \tag{2.4.20}$$

Since

$$|x-x_j| \geq \frac{d}{2}, \quad \forall x \in B_\varepsilon(x_i), \quad \forall j \neq i, \quad \forall i,j \in \{1,\dots,n\}, \tag{2.4.21}$$

from (2.4.20) we deduce that

$$\begin{aligned} I_1 &\leq \frac{\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N^2 (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \sum_{i=1}^n \int_{B_\varepsilon(x_i)\setminus B_{\varepsilon^2}(x_i)} |x-x_i|^{2(2-N)/n-2} dx \\ &= \frac{n\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N^2 (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} r^{N-1} \int_{S^{N-1}} r^{2(2-N)/n-2} d\sigma dr \\ &= \frac{n\left(\frac{d}{2}\right)^{2(n-1)(2-N)/n}}{\omega_N (N-2)^2} \frac{1}{\log^2(1/\varepsilon)} \int_{\varepsilon^2}^{\varepsilon} r^{(N-2)(1-2/n)-1} dr. \end{aligned}\tag{2.4.22}$$

From (2.4.20) we obtain that

$$I_1 = \begin{cases} O\big(\frac{1}{\log(1/\varepsilon)}\big), & n=2, \\ O(\varepsilon^{(N-2)(1-2/n)}), & n \geq 3. \end{cases}\tag{2.4.23}$$

Taking $\varepsilon > 0$ small enough such that $\varepsilon < 1/2m$, where $m = \max_{i=1,\dots,n} |x_i|$, it holds

$$|x-x_i| \geq \frac{1}{2\varepsilon}, \quad \forall x \in B_{2/\varepsilon}(0) \setminus B_{1/\varepsilon}(0),\ \forall i \in \{1,\dots,n\}. \tag{2.4.24}$$

Due to (2.4.24) we obtain

$$\begin{aligned} I_2 &= \frac{1}{\omega_N^2(N-2)^2} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^2 \prod_{i=1}^n |x-x_i|^{2(2-N)/n} dx \\ &\leq \frac{1}{\omega_N^2(N-2)^2} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^2 \prod_{i=1}^n \big(\frac{1}{2\varepsilon}\big)^{2(2-N)/n} dx \\ &= \frac{1}{\omega_N^2(N-2)^2} \big(\frac{1}{2}\big)^{2(2-N)} \int_{B_{2/\varepsilon}(0)\setminus B_{1/\varepsilon}(0)} \varepsilon^{2(N-1)} dx \\ &= \frac{2^{2(N-2)}}{\omega_N(N-2)^2} \varepsilon^{2(N-1)} \int_{1/\varepsilon}^{2/\varepsilon} r^{N-1} dr \\ &= O(\varepsilon^{N-2}), \quad \text{as } \varepsilon \to 0. \end{aligned}\tag{2.4.25}$$

In conclusion, according to (2.4.23) and (2.4.25) we get

$$\lim_{\varepsilon \searrow 0} \int |\nabla(u_\varepsilon E^{-1/n})|^2 E^{2/n} dx = 0, \quad \forall\, n \geq 2. \tag{2.4.26}$$

Combining (2.4.15), (2.4.18) and (3.3.14) we end up with the optimality of $(N-2)^2/n^2$ as in (2.4.17), and the proof of Theorem 2.4.1 is complete.

$\square$

Remark 2.4.1.

Our optimal result in Theorem 2.4.1 provides an inequality with a positive singular quadratic potential which behaves asymptotically like

$$\frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n}^{n} V_{ij}(x) \sim \frac{(N-2)^2}{4} \frac{2n-2}{n^2} \frac{1}{|x-x_i|^2} \qquad \text{as } x \to x_i, \quad \forall i \in \{1, \dots, n\}, \tag{2.4.27}$$

at each pole x_i. In particular, for $n \geq 3$, Theorem 2.4.1 represents an improvement of (2.4.3), *in the sense that the multiplication factor in* (2.4.27) *which corresponds to the quadratic singularity is larger than that one obtained in inequality* (2.4.3) *as emphasized in* (5.3.18).

On the contrary, for $n = 2$, the singularity in (2.4.3) *is asymptotically larger than the one produced in* (2.4.27).

Remark 2.4.2.

The proof of (2.4.3) *in [19] was obtained by expanding the square*

$$\int \Big| \nabla u + \alpha \sum_{i=1}^{n} \frac{x-x_i}{|x-x_i|^2} u \Big|^2 dx \geq 0, \quad \alpha \in \mathbb{R}, \tag{2.4.28}$$

which gives

$$0 \leq \int |\nabla u|^2 dx + [n\alpha^2 - (N-2)\alpha] \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx - \alpha^2 \sum_{1 \leq i < j \leq n} \int \frac{|x_i - x_j|^2}{|x-x_i|^2 |x-x_j|^2} u^2 dx. \tag{2.4.29}$$

More precisely, (2.4.3) *is a consequence of* (2.4.29) *when $\alpha = (N-2)/(2n)$. Moreover, we remark that the expansion* (2.4.29) *also applies to derive the inequality of Theorem 2.4.1 with a different choice of α, that is $\alpha = (N-2)/n$.*

The quadratic term in (2.4.14) *is given by the formula*

$$\int \Big| \nabla u - \frac{\nabla E^{1/n}}{E^{1/n}} u \Big|^2 dx = \int \Big| \nabla u + \frac{N-2}{n} \sum_{i=1}^{n} \frac{x-x_0}{|x-x_0|^2} u \Big|^2 dx,$$

which motivates the use of the "expansion of the square" emphasized above for $\alpha = (N-2)/n$. This was not observed in [19]. In fact we got to this point indirectly as a consequence of the direct application of identity (7.0.8).

Remark 2.4.3.

Adimurthi et al. proved in particular in [5] that, whenever E *satisfies* $-\Delta E = \sum_{1\leq i\leq n} \delta_{x_i}$ *for some given poles* $x_1, \dots, x_n \in \mathbb{R}^N$, *the following inequality holds*

$$\int |\nabla u|^2 dx \geq \frac{1}{4} \int \Big|\frac{\nabla E}{E}\Big| u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{2.4.30}$$

A direct application of (2.4.30) *in the context of multipolar Hardy inequalities would consist on taking* $E = E_1 + \ldots + E_n$. *If* $N \geq 3$ *we then get (see [2])*

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int \Big| \frac{\sum_{i=1}^n (x-x_i)|x-x_i|^{-N}}{\sum_{i=1}^n |x-x_i|^{2-N}} \Big|^2 u^2 dx, \quad \forall u \in H^1(\mathbb{R}^N). \tag{2.4.31}$$

Observe that the potential V *in* (2.4.31), *given by*

$$V_{N,n,x_1,\dots,x_n}(x) := \Big| \frac{\sum_{i=1}^n (x-x_i)|x-x_i|^{-N}}{\sum_{i=1}^n |x-x_i|^{2-N}} \Big|^2, \tag{2.4.32}$$

is non-negative and moreover has a quadratic singularity at each pole x_i. *More precisely,* $V_{N,n,x_1,\dots,x_n}$ *satisfies*

$$V_{N,n,x_1,\dots,x_n}(x) = \frac{1}{|x-x_i|^2} + O(|x-x_i|^{N-4}), \ \text{as } x \to x_i, \quad \forall i \in \{1, \dots, n\}, \tag{2.4.33}$$

respectively

$$V_{N,n,x_1,\dots,x_n}(x) = O(\frac{1}{|x|^2}), \quad \text{as } |x| \to \infty. \tag{2.4.34}$$

From (2.4.33) *and* (2.4.34) *we can easily deduce that*

$$V_{N,n,x_1,\dots,x_n}(x) = \sum_{i=1}^n \frac{1}{|x-x_i|^2} + O(1), \quad \forall x \in \mathbb{R}^N, \tag{2.4.35}$$

where $O(1)$ *denotes a changing sign quantity, uniformly bounded in* $\mathbb{R}^N$. *For* $N \geq 4$, *the identification* (2.4.35) *shows that inequality* (2.4.31) *allows to deduce an inequality in the spirit of* (2.4.2) *in which the same critical singular potential is obtained, paying the prize of adding a lower order term in* L^2*-norm. The multiplication factor of the lower order term obtained through* (2.4.31), *remains to be compared with that one which corresponds to* (2.4.2).

On the contrary, inequality (2.4.31) *does not allow to get optimal results as in Theorem 2.4.1 when removing the corrected lower order terms in* L^2*-norm.*

We point out, that the key role for showing Theorem 2.4.1 was played by identity (7.0.8) *applying for suitable distributions involving the product of the fundamental solutions of the Laplacian at each singular pole* x_i. *This allows to prove optimal Hardy inequalities for singular quadratic potentials of the form*

$$V_1(x) = \sum_{i=1}^n \frac{\lambda_i(x)}{|x-x_i|^2}, \quad \forall x \in \mathbb{R}^N,$$

where

$$\lambda_i(x) > 0 \text{ in } \mathbb{R}^N, \quad \lim_{x\to x_i} \lambda_i(x) = (n-1)\frac{(N-2)^2}{2n^2}, \quad \forall i \in \{1, \dots, n\}.$$

The weights λ_i in Theorem 2.4.1 are given by $\lambda_i(x) = (N-2)^2/(2n^2)\sum_{\substack{j=1\\ j\neq i}}^{n} |x_i-x_j|^2/|x-x_j|^2$.

As we mentioned before, the potential $V_{N,n,x_1,\dots x_n}$ cannot be compared with V_1 on any bounded connected domain Ω with $x_1,\dots,x_n \in \overline{\Omega}$. Indeed, next we emphasize this in two concrete examples.

Firstly, for $N=3$, $n=2$, we consider the singular poles $0, x_0 \in \overline{\Omega} \subset \mathbb{R}^3$ and we obtain $V_{3,2,0,x_0}(x_0/2) = 0$ while $V_1(x_0/2) > 0$.

Secondly, let us we consider for a configuration with three singular poles $x_1, x_2, x_3 \in \mathbb{R}^3$ determining an equilateral triangle such that

$$|x_1| = |x_2| = |x_3| > 0, \quad x_1 + x_2 + x_3 = 0,$$

and let $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with $x_1, x_2, x_3 \in \overline{\Omega}$. Then $V_{3,3,x_1,x_2,x_3}(0) = 0$ while $V_1(0) > 0$.

2.4.2. New bounds for the bipolar case

We now present some consequences of the previous multipolar Hardy inequality in Theorem 2.4.1 to bounded domains in $H_0^1(\Omega)$.

In this subsection we present some applications of Theorem 2.4.1 to bounded domains in the case of a bipolar potential

$$V(x) = \frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}, \tag{2.4.36}$$

for some $x_1, x_2 \in \mathbb{R}^N$, $N \geq 3$ with $x_1 \neq x_2$. In consequence, we derive new lower bounds for the bipolar Hardy inequality, which turn out to be optimal in the case where the poles are located on the boundary of the domain.

We have seen that Theorem 2.4.1 provides an inequality involving a bipolar potential V_1 which behaves asymptotically like

$$V_1(x) \sim \frac{(N-2)^2}{8}\frac{1}{|x-x_i|^2}, \quad \text{as } x \to x_i, \quad \forall i \in \{1,2\}. \tag{2.4.37}$$

On the other hand, inequality (2.4.3) provides a bipolar potential with a stronger quadratic singularity which is asymptotically given by

$$V_2(x) \sim \frac{5(N-2)^2}{32}\frac{1}{|x-x_i|^2}, \quad \text{as } x \to x_i, \quad \forall i \in \{1,2\}. \tag{2.4.38}$$

Surprisingly, Theorem 2.4.1 may give better lower bounds than inequality (2.4.3) for the Hardy inequality with the bipolar potential V as in (2.4.36). The main results of this section are as follows.

As a consequence of Theorem 2.4.1 we have

Proposition 2.4.1. *Assume $0 \leq \alpha, \beta \leq 1$. For any $x_1 \neq x_2$ and $u \in C_0^\infty(B_{r(x_1,x_2)}(C(x_1,x_2)))$ we have*

$$\int_{B_{r(x_1,x_2)}(C(x_1,x_2))} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4}\int_{B_{r(x_1,x_2)}(C(x_1,x_2))} \Big[\frac{\alpha}{|x-x_1|^2} + \frac{\beta}{|x-x_2|^2}\Big] u^2 dx, \tag{2.4.39}$$

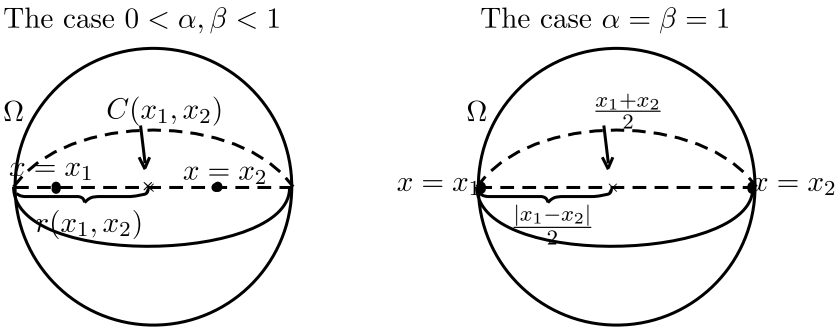

Figure 2.3: Domains where improved bipolar inequalities hold. The best results are obtained when both singularities are located on the boundary, case which corresponds to $\alpha=\beta=1$.

where $B_{r(x_1,x_2)}(C(x_1,x_2))$ *is the ball centered at the point*

$$C(x_1,x_2)=\frac{\beta}{\alpha+\beta}x_1+\frac{\alpha}{\alpha+\beta}x_2,$$

of radius

$$r(x_1,x_2)=\frac{\sqrt{\alpha+\beta-\alpha\beta}}{\alpha+\beta}|x_1-x_2|,$$

as shown in Figure 2.3.

As a consequence of inequality (2.4.3) we have

Proposition 2.4.2. *Assume* $0\leq\alpha,\beta\leq 1$. *For any* $x_1\neq x_2$ *and* $u\in C_0^\infty(B_{r(x_1,x_2)}(C(x_1,x_2)))$ *we have*

$$\int_{B_{r(x_1,x_2)}(C(x_1,x_2))}|\nabla u|^2dx\geq\Big[\frac{(N-2)^2}{8}+\frac{(N-2)^2}{16}\alpha\Big]\int_{B_{r(x_1,x_2)}(C(x_1,x_2))}\frac{u^2}{|x-x_1|^2}dx$$
$$+\Big[\frac{(N-2)^2}{8}+\frac{(N-2)^2}{16}\beta\Big]\int_{B_{r(x_1,x_2)}(C(x_1,x_2))}\frac{u^2}{|x-x_2|^2}dx, \tag{2.4.40}$$

where $B_{r(x_1,x_2)}(C(x_1,x_2))$ *is the ball centered at the point*

$$C(x_1,x_2)=\frac{\beta}{\alpha+\beta}x_1+\frac{\alpha}{\alpha+\beta}x_2,$$

of radius

$$r(x_1,x_2)=\frac{\sqrt{\alpha+\beta-\alpha\beta}}{\alpha+\beta}|x_1-x_2|,$$

as shown in Figure 2.3.

Remark 2.4.4. *The constraints* $\alpha,\beta\leq 1$ *impose to the singular poles* x_1,x_2 *to belong to* $\in\overline{B_{r(x_1,x_2)}(C(x_1,x_2))}$.

Remark 2.4.5. *We observe that for α, β getting closer to 1, the result of Proposition 2.4.1 is better than the one of Proposition 2.4.2.*

Next we prove only Proposition 2.4.1 since the proof of Proposition 2.4.2 follows the same steps.

Proof of Proposition 2.4.1.

Let us consider an open bounded subset $\Omega \subset \mathbb{R}^N$, $N \geq 3$. Applying Theorem 2.4.1 we have that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_\Omega \Big| \frac{x - x_1}{|x - x_1|^2} - \frac{x - x_2}{|x - x_2|^2} \Big|^2 dx, \forall u \in H_0^1(\Omega). \tag{2.4.41}$$

In the sequel, we are seeking for domains $\Omega \subset \mathbb{R}^N$ such that $x_1, x_2 \in \overline{\Omega}$ and

$$\Big| \frac{x - x_1}{|x - x_1|^2} - \frac{(x - x_2)}{|x - x_2|^2} \Big|^2 \geq \frac{\alpha}{|x - x_1|^2} + \frac{\beta}{|x - x_2|^2}, \quad \forall x \in \overline{\Omega}. \tag{2.4.42}$$

Using the identity $2(x - x_1)(x - x_2) = |x - x_1|^2 + |x - x_2|^2 - |x_1 - x_2|^2$, then (2.4.42) is equivalent to

$$|x_1 - x_2|^2 \geq \alpha |x - x_2|^2 + \beta |x - x_1|^2, \quad \forall x \in \overline{\Omega}. \tag{2.4.43}$$

Expanding the squares in (2.4.43) and dividing by $\alpha + \beta$ we obtain

$$\frac{1-\beta}{\alpha+\beta}|x_1|^2 + \frac{1-\alpha}{\alpha+\beta}|x_2|^2 - \frac{2}{\alpha+\beta} x_1 \cdot x_2 \geq |x|^2 - 2x \cdot \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big). \tag{2.4.44}$$

Coupling the squares we rewrite (2.4.44) as

$$\frac{1-\beta}{\alpha+\beta}|x_1|^2 + \frac{1-\alpha}{\alpha+\beta}|x_2|^2 - \frac{2}{\alpha+\beta} x_1 \cdot x_2 + \Big| \frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big|^2 \geq \Big| x - \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big) \Big|^2. \tag{2.4.45}$$

After some computations on the left hand side of (2.4.45) we reaches

$$\frac{\alpha + \beta - \alpha\beta}{(\alpha+\beta)^2} |x_1 - x_2|^2 \geq \Big| x - \Big(\frac{\alpha}{\alpha+\beta} x_2 + \frac{\beta}{\alpha+\beta} x_1 \Big) \Big|^2, \quad \forall x \in \overline{\Omega}. \tag{2.4.46}$$

Due to this, the proof is finished by identifying properly the set Ω.

□

We notice that, as far as α and β get closer to 1, the poles $x = x_1$ respectively $x = x_2$, are pushed to the boundary of the domain as drawn in Figure 2.3. Indeed, if $\alpha = \beta = 1$ then x_1 and x_2 are located on the boundary of $B_{r(x_1,x_2)}(C(x_1, x_2)) = B_{|x_1 - x_2|/2}((x_1 + x_2)/2)$. Moreover, we will have the non trivial inequality

$$\int_{B_{\frac{|x_1 - x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 \geq \frac{(N-2)^2}{4} \int_{B_{\frac{|x_1 - x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \Big[\frac{1}{|x - x_1|^2} + \frac{1}{|x - x_2|^2} \Big] u^2 dx. \tag{2.4.47}$$

for all $u \in C_0^1(B_{|x_1 - x_2|/2}((x_1 + x_2)/2))$.

As we said before, inequality (2.4.47) is new and not trivial. In fact, it provides an improved result in higher dimensions as follows.

Applying Hardy inequalities with boundary singularities (see e.g. [38]) we have that

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \frac{u^2}{|x-x_1|^2} dx,$$

and

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 dx \geq \frac{N^2}{4} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \frac{u^2}{|x-x_2|^2} dx,$$

the constant $N^2/4$ being optimal in both cases. Thus

$$\int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} |\nabla u|^2 \geq \frac{N^2}{8} \int_{B_{\frac{|x_1-x_2|}{2}}\left(\frac{x_1+x_2}{2}\right)} \Big[\frac{1}{|x-x_1|^2} + \frac{1}{|x-x_2|^2}\Big] u^2 dx.$$

Note that inequality (2.4.47) is better for $N \geq 7$ since

$$\frac{(N-2)^2}{4} \geq \frac{N^2}{8}, \quad \forall\, N \geq 7.$$

2.5. Inequalities with an infinite number of singularities

To the best of our knowledge, the case of infinite number of singularities has been very less investigated in the literature. The only related result we have seen refers to the paper [63] in which the authors showed that for any $\lambda < (N-2)^2/4$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ satisfying

$$\sum_{n=1}^{\infty} |a_n|^{-(N-2)} < \infty, \quad \sum_{k=1}^{\infty} |a_{n+k} - a_n|^{-(N-2)} \text{ is bounded uniformly in } n,$$

$|a_n - a_m| \geq 1$ for all $n \neq m$, there exists $\delta > 0$ such that

$$\forall u \in H^1(\mathbb{R}^N), \quad \int |\nabla u|^2 dx - \lambda \sum_{n=1}^{\infty} \int \frac{\chi_{B_\delta(a_n)} u^2}{|x-a_n|^2} dx \geq 0. \tag{2.5.1}$$

In this section we give some positive results for potentials with infinite number of singularities periodically distributed on a line in $\mathbb{R}^N$, $N \geq 2$. We point out that when $N = 3$ our sequence of singularities violates the hypotheses of $\{a_n\}_{n\in\mathbb{N}}$ above and therefore (2.5.1) does not apply.

To be more precise, for any $N \geq 2$ we define the line

$$l := \{x \in \mathbb{R}^N \mid x = \vec{a} + t\vec{h}, \quad t \in \mathbb{R}\}, \tag{2.5.2}$$

where $\vec{a}$, $\vec{h}$ are fixed vectors in $\mathbb{R}^N$. For any $R > 0$ and any line l introduced in (2.5.2), we associate the cylinder

$$\mathcal{C}_{\{R,l\}} := \{x \in \mathbb{R}^N \mid d(x,l) \leq R\}. \tag{2.5.3}$$

where $d(x,l)$ denotes the euclidian distance from a current point $x \in \mathbb{R}^N$ to the line l. Moreover, we consider the family of points $\{p_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}^N$ defined periodically on l, i.e. $p_i := \vec{a} + i\vec{h}$. Next we define the period by

$$h := |\vec{h}| = |p_{i+1} - p_i| > 0, \quad \forall\, i \in \mathbb{Z}, \tag{2.5.4}$$

and we call the sequence $\{p_i\}_{i\in\mathbb{Z}}$ to be *h-periodic*.

2.5.1. Main results

The main result of this Section is stated as follows.

Theorem 2.5.1. *Assume $N \geq 2$. Let us consider a cylinder $\mathcal{C}_{R,l}$ with an h-periodic sequence of singularities $\{p_i\}_{i\in\mathbb{Z}}$. Then, the following multipolar Hardy inequalities hold*

$$\int_{\mathcal{C}_{R,l}} |\nabla u|^2 dx \geq \begin{cases} \frac{h^2}{4(R^2\pi^2/3+2h^2)} \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{R,l}} \frac{u^2}{|x-p_i|^2} dx, & \forall\, u \in H_0^1(\mathcal{C}_{R,l} \setminus l), N = 2, \\ \frac{h^2}{8(R^2\pi^2/3+2h^2)} \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{R,l}} \frac{u^2}{|x-p_i|^2} dx, & \forall\, u \in H_0^1(\mathcal{C}_{R,l}),\ N = 3, \\ \left(\frac{N-3}{2}\right)^2 \frac{h^2}{(R^2\pi^2/3+2h^2)} \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{R,l}} \frac{u^2}{|x-p_i|^2} dx, & \forall u \in H_0^1(\mathcal{C}_{R,l}),\ N \geq 4. \end{cases} \tag{2.5.5}$$

Firstly, observe that the results of Theorem 3.3.1 are invariant with respect to any cylinder $\mathcal{C}_{R,l}$ with an h-periodic sequence of singularities $\{p_i\}_{i\in\mathbb{Z}}$ located on l. Therefore, without losing the generality, it is enough to prove Theorem 3.3.1 for the N-d line

$$l_N := \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_2 = x_3 = \ldots = x_N = 0\}.$$

and the associated cylinder

$$\mathcal{C}_{R,l_N} := \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_2^2 + \ldots x_N^2 < R^2\},$$

in which the singularities $\{p_i\}_{i\in\mathbb{Z}}$ are given by

$$p_i = ihe_1, \quad \forall i \in \mathbb{Z}, \quad e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N.$$

Then Theorem 3.3.1 is a direct consequence of Remark 2.5.1 combined with Propositions 2.5.1, 2.5.2 which are stated as follows.

Proposition 2.5.1. *Assume $N = 2$ or $N \geq 4$. We claim that for all $u \in C_0^1(\mathcal{C}_{R,l_N} \setminus l_N)$ it holds the inequality*

$$\int_{\mathcal{C}_{R,l_N}} |\nabla u|^2 dx \geq \Big(\frac{N-3}{2}\Big)^2 \frac{h^2}{(R^2\pi^2/3 + 2h^2)} \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{R,l_N}} \frac{u^2}{|x_1 - ih|^2 + x_2^2 + \ldots + x_N^2} dx. \tag{2.5.6}$$

Proposition 2.5.2. *Assume $N = 3$. We claim that for all $u \in C_0^1(\mathcal{C}_{R,l_3} \setminus l_3)$ the following inequality is satisfied:*

$$\int_{\mathcal{C}_{R,l_3}} |\nabla u|^2 dx \geq \frac{h^2}{2(R^2\pi^2/3 + 2h^2)} \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{R,l_3}} \frac{u^2}{|x_1 - ih|^2 + x_2^2 + x_3^2} dx. \tag{2.5.7}$$

Remark 2.5.1. *(see [56]) For any $N \geq 3$ and $R > 0$, since l_N has zero capacity it holds that $C_0^1(\mathcal{C}_{R,l_N} \setminus l_N)$ is dense in $H_0^1(\mathcal{C}_{R,l_N})$.*

2.5.2. Proof of Proposition 2.5.1

Proposition 2.5.1 is proved in the following. Then we sketch the proof of Proposition 2.5.2 since it follows the same strategy as Proposition 2.5.1.

Proof of Proposition 2.5.1.

We are interested in finding $\lambda > 0$ such that $L = -\Delta - \lambda V(x)$ is positive in the L^2-quadratic form, where

$$V(x) = \sum_{i \in \mathbb{Z}} \frac{1}{(|x_1 - ih|^2 + x_2^2 + \ldots + x_N^2)}.$$

Firstly, observe that,

$$V(x + he_1) = V(x), \quad \forall x \in \mathbb{R}^N \text{ and } e = (1, 0, \ldots, 0) \in \mathbb{R}^N. \tag{2.5.8}$$

Moreover,

$$V(p_i) = \infty, \quad \forall\ x_i = ihe_1 \in l_N,\ i \in \mathbb{Z}.$$

Next let us fix $u \in C_0^1(\Omega \setminus l_N)$ and, in the spirit of (7.0.8) we consider the transformation

$$u = \phi v, \quad \phi > 0 \text{ in } \mathcal{C}_{R,l_N} \setminus l_N; \quad \phi(x) = 0,\ \forall x \in l_N. \tag{2.5.9}$$

Applying (7.0.8) (integrations by parts are allowed since there is enough regularity for the functions involved in the computations) we get

$$\int |\nabla u|^2 dx = \int |\nabla v|^2 \phi^2 dx - \int \frac{\Delta \phi}{\phi} u^2 dx, \tag{2.5.10}$$

and in particular it holds that

$$\int |\nabla u|^2 dx \geq \int \Big(- \frac{\Delta \phi}{\phi} \Big) u^2 dx. \tag{2.5.11}$$

Inspired by the fact that all the singularities are located on the same line, we will choose the family of functions ϕ_α such that $\phi_\alpha(x + he_1) = \phi_\alpha(x)$ for all $x \in \mathcal{C}_{R,l_N}$, that is

$$\phi_\alpha(x) = (x_2^2 + x_3^2 + \ldots + x_N^2)^\alpha, \tag{2.5.12}$$

where $\alpha > 0$ will be precise later (in such way that it maximizes $-\Delta\phi_\alpha/\phi_\alpha$). Note that ϕ_α fulfills the conditions in (2.5.9). Indeed, note that we directly obtain

$$-\frac{\Delta \phi_\alpha}{\phi_\alpha} = \frac{-4\alpha^2 - 2\alpha(N-3)}{(x_2^2 + x_3^2 + \ldots + x_N^2)^2}, \quad \forall x \in \mathcal{C}_{R,l_N} \setminus l_N.$$

We choose $\alpha_\star = -(N-3)/4$ which maximizes

$$\max_\alpha \{-4\alpha^2 - 2\alpha(N-3)\} = \Big(\frac{N-3}{2}\Big)^2.$$

Then we have

$$-\frac{\Delta\phi_{\alpha_\star}}{\phi_{\alpha_\star}} = \Big(\frac{N-3}{2}\Big)^2 \frac{1}{(x_2^2+x_3^2\ldots x_N^2)^2}, \quad \forall x \in \mathcal{C}_{R,l_N}\setminus l_N. \tag{2.5.13}$$

In the sequel, we will show that

$$\frac{1}{(x_2^2+x_3^2+\ldots x_N^2)^2} \geq \frac{h^2}{R^2\pi^2/3+2h^2}\sum_{i\in\mathbb{Z}}\frac{1}{|x_1-ih|^2+x_2^2+\ldots+x_N^2}, \quad \forall x\in\mathcal{C}_{R,l_N}\setminus l_N, \tag{2.5.14}$$

which combined with (2.5.13) we get

$$-\frac{\Delta\phi_{\alpha_\star}}{\phi_{\alpha_\star}} \geq \Big(\frac{N-3}{2}\Big)^2\frac{h^2}{R^2\pi^2/3+2h^2}V(x), \quad \forall x\in\mathcal{C}_{R,l_N}\setminus l_N, \tag{2.5.15}$$

and according to (2.5.11), the proof of Proposition 2.5.1 finishes.

In order to end up with (2.5.14), let us firstly define $\Omega_{R,k,h}=\{x\in\mathcal{C}_{R,l_N}\mid kh\leq x_1\leq (k+1)h\}$ and notice that $\mathcal{C}_{R,l_N}=\cup_{k\in\mathbb{Z}}\Omega_{R,k,h}$. Next, it is easy to see that for any fixed $k\in\mathbb{Z}$ we have

$$\begin{cases} |x_1-ih|^2+x_2^2+\ldots+x_N^2\geq x_2^2+\ldots+x_N^2, & i=k,k+1,\ \forall x\in\Omega_{R,k,h},\\ |x_1-ih|^2\geq h^2|i-k|^2\geq \frac{h^2}{R^2}|i-k|^2(x_2^2+\ldots+x_N^2), & i<k,\ \forall x\in\Omega_{R,k,h},\\ |x_1-ih|^2\geq h^2|i-(k+1)|^2\geq |i-(k+1)|^2\frac{h^2}{R^2}(x_2^2+\ldots+x_N^2), & i>k+1,\ \forall x\in\Omega_{R,k,h}. \end{cases} \tag{2.5.16}$$

Then for any $x\in\Omega_{R,k,h}\setminus l_N$, according to (2.5.16) we obtain

$$\begin{aligned} \sum_{i\in\mathbb{Z}}\frac{1}{|x_1-ih|^2+x_2^2+\ldots+x_N^2} &\leq \frac{1}{x_2^2+\ldots+x_N^2}\Big(\sum_{i<k}\frac{R^2}{h^2|k-i|^2}+\sum_{i>k+1}\frac{R^2}{h^2|k+1-i|^2}+\sum_{i=k,k+1}1\Big)\\ &= \frac{1}{x_2^2+\ldots+x_N^2}\Big(\frac{R^2}{h^2}\sum_{j\geq1}\frac{1}{j^2}+\frac{R^2}{h^2}\sum_{j\geq1}\frac{1}{j^2}+2\Big)\\ &= \frac{1}{x_2^2+\ldots+x_N^2}\Big(\frac{2R^2}{h^2}\sum_{j\geq1}\frac{1}{j^2}+2\Big)\\ &= \frac{1}{x_2^2+\ldots+x_N^2}\Big(\frac{R^2\pi^2}{3h^2}+2\Big). \end{aligned} \tag{2.5.17}$$

Therefore

$$\frac{1}{x_2^2+\ldots x_N^2}\geq\frac{h^2}{R^2\pi^2/3+2h^2}\sum_{i\in\mathbb{Z}}\frac{1}{|x_1-ih|^2+x_2^2+\ldots x_N^2}, \quad \forall x\in\mathcal{C}_{R,l_N}\setminus l_N,$$

which finishes the proof of Proposition 2.5.1.

□

Proof of Proposition 2.5.2.

Let $u \in C_0^1(\mathcal{C}_{R,l_3} \setminus l_3)$ be fixed and consider the transformation

$$u = \phi v, \quad \phi > 0 \text{ in } \mathcal{C}_{R,l_3} \setminus d_3; \quad \phi(x) = 0, \quad \forall x \in l_3. \tag{2.5.18}$$

As in Proposition 2.5.1 we obtain

$$\int |\nabla u|^2 dx \geq \int \Big(-\frac{\Delta \phi}{\phi} \Big) u^2 dx. \tag{2.5.19}$$

Next we consider

$$\phi_\alpha(x) = |x_2|^\alpha + |x_3|^\alpha, \tag{2.5.20}$$

for some constant $\alpha \in \mathbb{R}$ aimed to be determined. Note that ϕ_α fulfills the conditions in (2.5.18). After direct computations we obtain

$$-\frac{\Delta \phi_\alpha}{\phi_\alpha} = \frac{(-\alpha^2 + \alpha)(|x_2|^{\alpha-2} + |x_3|^{\alpha-2})}{|x_2|^\alpha + |x_3|^\alpha}, \quad \forall x \in \mathcal{C}_{R,l_3} \setminus l_3. \tag{2.5.21}$$

Next we choose $\alpha = 1/2$ which maximizes

$$\max_\alpha \{-\alpha^2 + \alpha\} = \frac{1}{4}.$$

In consequence, we have

$$-\frac{\Delta \phi_{1/2}}{\phi_{1/2}} = \frac{1}{4} \frac{|x_2|^{-3/2} + |x_3|^{-3/2}}{|x_2|^{1/2} + |x_3|^{1/2}}. \tag{2.5.22}$$

Using the Cebyshev inequality we have

$$(|x_2|^{-3/2} + |x_3|^{-3/2})(x_2^2 + x_3^2) \geq 2(|x_2|^{1/2} + |x_3|^{1/2}).$$

It follows that

$$-\frac{\Delta \phi_{1/2}}{\phi_{1/2}} \geq \frac{1}{2(x_2^2 + x_3^2)}, \quad \forall x \in \mathcal{C}_{R,l_3} \setminus l_3. \tag{2.5.23}$$

According to (2.5.19), (2.5.23) and the inequality

$$\frac{1}{x_2^2 + x_3^2} \geq \frac{h^2}{R^2\pi^2/3 + 2h^2} \sum_{i \in \mathbb{Z}} \frac{1}{|x_1 - ih|^2 + x_2^2 + x_3^2}, \quad \forall x \in \mathcal{C}_{R,l_3} \setminus l_3,$$

proved in Proposition 2.5.1, we finish the proof of Proposition 2.5.2.

$\square$

Chapter 3

Pohozaev identity and applications to the control of waves

Abstract. The aim of this chapter is two folded. Firstly, we study the validity of a Pohozaev-type identity for the Schrödinger operator

$$A_\lambda := -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda \in \mathbb{R},$$

in the situation where the origin is located on the boundary of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, showing some applications to semi-linear elliptic equations. The problem we address is very much related to optimal Hardy-Poincaré inequalities with boundary singularities which have been investigated in the recent past in various papers. In view of that, the proper functional framework is described and explained.

Secondly, we use the Pohozaev identity to derive the method of multipliers and we apply it to study the exact boundary controllability for the wave and Schrödinger equations corresponding to the singular operator A_λ. In particular, this complements and extends well known results by Vanconstenoble and Zuazua [109], who discussed the same issue in the case of interior singularity.

3.1. Introduction

In this chapter we are dealing with the Schrödinger operator $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, acting in a domain where the potential $1/|x|^2$ is singular at the boundary. Our main goal consists in studying the control properties of the corresponding wave and Schrödinger equations. Moreover, our aim is to find necessary and sufficient conditions for the existence of non-trivial solutions to semi-linear elliptic equations associated to A_λ. Operators like A_λ may arise in molecular physics [86], quantum cosmology [14], combustion models [71], linearization of critical nonlinear PDE's (e.g. [26], [104]), etc... From a mathematical point of view they are interesting due to their criticality since they are homogeneous of degree -2.

The qualitative properties of evolution problems involving the operator A_λ require either positivity or coercivity of A_λ in the sense of quadratic forms in L^2. Roughly speaking, this is equivalent to making use of Hardy-type inequalities. Let us recall some of the inequalities discussed in Chapter 2 which will be useful in the following. Assume Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, containing the origin, i.e., $0 \in \Omega$. Then it holds (see [78])

$$\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx > 0, \quad \forall u \in H_0^1(\Omega), \tag{3.1.1}$$

and the constant $(N-2)^2/4$ is optimal and not attained in $H_0^1(\Omega)$. We remind that the optimal Hardy constant is defined by the quotient

$$\mu(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega \frac{u^2}{|x|^2} dx \Big).$$

In this chapter, we consider Ω to be a smooth subset of $\mathbb{R}^N$, $N \geq 1$, with the origin $x = 0$ placed on its boundary Γ.

Starting with the work by Filippas, Tertikas and Tidblom [67], and continuing with [38], [58], [57], [59], it has been proved that, whenever Ω is a smooth domain with the origin located on the boundary, there exists a positive constant $r_0 = r_0(\Omega, N) > 0$ such that

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}. \tag{3.1.2}$$

where $B_{r_0}(0) \subset \mathbb{R}^N$ denotes the ball of radius r_0 centered in origin. Next we recall the definition of the upper half space $\mathbb{R}^N_+$ which is given by the set

$$\mathbb{R}^N_+ = \big\{ x = (x_1, \ldots, x_N) = x' + x_N e_N \in \mathbb{R}^N \mid x_N > 0 \big\}, \tag{3.1.3}$$

where e_N is the N-th canonical vector in $\mathbb{R}^N$ and $x' = (x_1, \ldots, x_{N-1}, 0)$. In addition, if $\Omega \subset \mathbb{R}^N_+$, $N \geq 1$, the new Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx, \quad \forall u \in H_0^1(\Omega), \tag{3.1.4}$$

holds true and the constant $N^2/4$ is optimal, i.e. $\mu(\Omega) = N^2/4$.

Otherwise, if Ω is a smooth domain which, up to a rotation, is not supported in $\mathbb{R}^N_+$, the constant $N^2/4$ is optimal, up to lower order terms in $L^2(\Omega)$-norm as shown later in inequality (3.1.8). In general $\mu(\Omega) = N^2/4$ is not true for any smooth bounded domain Ω containing the origin on the boundary (e.g. [57]).

Without losing generality, since the operator A_λ is invariant under rotations, next in the chapter we consider Ω such that

$$x \cdot \nu = O(|x|^2), \text{ on } \Gamma, \tag{3.1.5}$$

where ν stands for the outward normal vector to Γ. Moreover, since optimal inequalities have been obtained regardless of the shape of Ω, throughout the chapter we discuss two main situations of geometries motivated by the remarks above.

C1. Ω is a smooth domain satisfying (3.1.5) and $x_N > 0$ holds for all $x \in \Omega$ (i.e. $\Omega \subset \mathbb{R}^N_+$).

C2. Ω is a smooth domain satisfying (3.1.5) such that x_N changes sign in Ω ($\Omega \not\subset \mathbb{R}^N_+$).

Next we need to introduce the constant

$$R_\Omega := \sup_{x\in\overline{\Omega}} |x|. \tag{3.1.6}$$

The following optimal Hardy-Poincaré inequalities are valid for each one of the cases above.

If Ω fulfills the case C1, then (e.g. [38]) it holds that

$$\forall\, u \in C_0^\infty(\Omega), \quad \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx + \frac{1}{4}\int_\Omega \frac{u^2}{|x|^2\log^2(R_\Omega/|x|)}dx, \tag{3.1.7}$$

and $N^2/4$ is the sharp constant.

If Ω satisfies the case C2 then (e.g. [57]) there exist two constants $C_2 = C_2(\Omega) \in \mathbb{R}$ and $C_3 = C_3(\Omega, N) > 0$ such that for any $u \in C_0^\infty(\Omega)$ it holds

$$C_2\int_\Omega u^2dx + \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{u^2}{|x|^2}dx + C_3\int_\Omega \frac{u^2}{|x|^2\log^2(R_\Omega/|x|)}dx. \tag{3.1.8}$$

Due to the inequalities mentioned above, a new notation is used throughout the chapter, namely

$$\lambda(N) := \frac{N^2}{4}. \tag{3.1.9}$$

In view of this, let us now describe the content of the chapter.

In Section 3.2 we introduce the functional framework induced by the above Hardy inequalities. We refer to the Hilbert space H_λ defined in Subsection 3.2.1. Then we check the validity of the Pohozaev identity for the Schrödinger operator A_λ in this functional setting. For that we define the domain of A_λ as

$$D(A_\lambda) := \big\{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\big\}, \tag{3.1.10}$$

and we prove that

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = -\int_\Omega (x\cdot\nabla u)A_\lambda u dx - \frac{N-2}{2}||u||^2_{H_\lambda}, \quad \forall u \in D(A_\lambda), \tag{3.1.11}$$

where $||\cdot||_{H_\lambda}$ denotes the norm associated to H_λ. We refer to Theorems 3.2.1, 3.2.2 for a complete statement of this result. For the sake of clarity, we will mainly discuss the case C1 above. Nevertheless, similar results could be also extended to the case C2 in a weaker functional setting due to weaker Hardy inequalities (see Subsection 3.2.2).

Formally, identity (3.1.11) can be obtained by direct integrations. However, a rigorous justification of the integrations is needed due to the lack of regularity of A_λ at the origin where standard elliptic regularity does not apply. In addition, we need to justify the integrability of the boundary term in (3.1.11) which is no more obvious since the singularity is located on the boundary and standard trace regularity fails. As we mentioned before, we give a rigorous justification of these facts in Theorems 3.2.1, 3.2.2.

Pohozaev type identities arise in many applications and mostly when studying non-linear equations (see [55], [73], [47] and references therein).

In Section 3.3, we apply Theorem 3.2.2 to characterize the existence of non-trivial solutions to a semi-linear singular elliptic PDE in star-shaped domains. We refer mainly to Theorem 3.3.1.

In Section 3.4 we present some applications of the Pohozaev identity in Theorem 3.2.2 to the controllability of conservative systems like wave and Schrödinger equations, for which the multiplier method plays a crucial role.

In the last few decades, most of the studies in Controllability Theory and its applications to evolution PDEs, have applied methods like *Hilbert Uniqueness Method* (HUM) introduced by J. L. Lions in [88], Carleman estimates developed by Fursikov and Imanuvilov [69], microlocal analysis due to Bardos, Lebeau and Rauch ([12], [11]), but also multiplier techniques with the pioneering papers by Komornik and Zuazua ([82], [83], [114]). In particular, the controllability properties and stabilization of the heat like equation corresponding to A_λ have been analyzed in [108], [54], [107] in the case of interior singularity using tools based on Carleman estimates.

Now, let us detail the controllability problem we are interested in Section 3.4. For $N \geq 1$ we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ where Γ denotes its boundary. Moreover, we denote by Γ_0 a non-empty part of the set Γ that will be specified later.

Next we consider the wave-like process

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{3.1.12}$$

To better specify the problem under consideration, we say that the system (3.1.12) is exactly controllable from Γ_0, in time T, if for any initial data $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and any target $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$, there exists a control $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of (3.1.12) satisfies:

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \text{for all } x \in \Omega.$$

This issue was analyzed by Vancostenoble and Zuazua [109] under the assumption that the singularity $x = 0$ is located in the interior of Ω. They proved well-posedness and exact controllability of system (3.1.12) for any $\lambda \leq \lambda_\star := (N-2)^2/4$ for boundary controls acting in Γ_0 defined by

$$\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}. \tag{3.1.13}$$

Roughly speaking, the authors showed in [109] that the parameter $\lambda_\star$ is critical when asking the well-posedness and control properties of (3.1.12), and the results are very much related to the best constant in the Hardy inequality with interior singularity.

In Section 3.4, we address the same controllability question in the case of boundary singularity. Our main result asserts that for the same geometrical setup (3.1.13), we can increase the range of values λ (from $\lambda_\star$ to $\lambda(N)$) for which the exact boundary controllability of system (3.1.12) holds. This is due to the new Hardy inequalities above.

By now classical HUM, the Controllability of system (3.1.12) is equivalent to the so-called

Observability Inequality for the adjoint system

$$\begin{cases} w_{tt} - \Delta w - \lambda \frac{w}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ w(t,x) = 0, & (t,x) \in (0,T) \times \Gamma, \\ w(0,x) = w_0(x), & x \in \Omega, \\ w_t(0,x) = w_1(x), & x \in \Omega, \end{cases} \tag{3.1.14}$$

which formally states that for any $\lambda \leq \lambda(N)$ and $T > 0$ large enough there exists a constant $C_T > 0$ such that

$$C_T \left(||w_1||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla w_0(x)|^2 - \lambda \frac{w_0^2(x)}{|x|^2} \right) dx \right) \leq \int_0^T \int_{\Gamma_0} (x \cdot \nu) \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt, \tag{3.1.15}$$

holds true for w solution of (3.1.14). We point out that, since the weight $x \cdot \nu$ degenerates at the origin, our inequality (3.1.15) is stronger than the one proved in the case of interior singularity in [109] which formally states that

$$C_T \left(||w_1||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla w_0(x)|^2 - \lambda \frac{w_0^2(x)}{|x|^2} \right) dx \right) \leq \int_0^T \int_{\Gamma_0} \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt. \tag{3.1.16}$$

The main tool to prove (3.1.15) relies on the multiplier method and compactness-uniqueness argument [88]. In view of that, the Pohozaev identity provides a direct tool to show that the solution of system (3.1.14) satisfies the multiplier identity which is formally given by

$$\frac{1}{2} \int_0^T \int_\Gamma (x \cdot \nu) \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt = \frac{T}{2} (||w_1||^2_{L^2(|\Omega)} + ||w_0||^2_{H_\lambda}) + \int_\Omega w_t \left(x \cdot \nabla w + \frac{N-1}{2} w \right) \Big|_{t=0}^{t=T} dx, \tag{3.1.17}$$

producing a “Hidden regularity” effect for the normal derivative. We refer to Theorem 3.4.2 for a rigorous statement. As a consequence, the solution of system (3.1.14) verifies the reverse Observability inequality. Then identity (3.1.17) together with the sharp-Hardy inequality stated in Theorem 3.1.1 lead to *Observability inequality* (3.1.15) as emphasized in Theorem 3.4.3.

Theorem 3.1.1. *Assume Ω satisfies one of the cases C1-C2 . Then, there exists a constant $C = C(\Omega) \in \mathbb{R}$ such that*

$$\int_\Omega |x|^2 |\nabla w|^2 dx \leq R_\Omega^2 \left(\int_\Omega |\nabla w|^2 dx - \frac{N^2}{4} \int_\Omega \frac{w^2}{|x|^2} dx \right) + C \int_\Omega w^2 dx, \quad \forall w \in C_0^\infty(\Omega). \tag{3.1.18}$$

Theorem 3.1.1 above, whose proof is given in the Appendix, extends to the case of boundary singularity a similar inequality shown in [109] ,on page 2, as part of Theorem 1.1, in the context of interior singularity.

Remark 3.1.1. *The result of Theorem 3.1.1, more precisely the constant R_Ω^2 which appears in inequality* (3.1.18)*, helps to obtain the control time $T > T_0 = 2R_\Omega$ in* (3.1.15)*, which is expected to be optimal due to the Geometric Control Condition (GCC), see e.g. [12].*

Although Theorem 3.1.1 is sharp for our applications to controllability, it is worth mentioning that we are able to obtain a more general result as follows.

Theorem 3.1.2. *Assume Ω satisfies one of the cases C1-C2. Let be $\varepsilon > 0$ small enough. Then, there exists a constant $C_\varepsilon = C(\Omega, \varepsilon) \in \mathbb{R}$ such that*

$$\int_\Omega |x|^\varepsilon |\nabla w|^2 dx \leq R_\Omega^\varepsilon \Big(\int_\Omega |\nabla w|^2 dx - \frac{N^2}{4} \int_\Omega \frac{w^2}{|x|^2} dx \Big) + C_\varepsilon \int_\Omega w^2 dx, \quad \forall w \in C_0^\infty(\Omega). \quad (3.1.19)$$

We have omitted including the proof of Theorem 3.1.2 since it applies the same steps as in the case of Theorem 3.1.1.

Finally in Section 3.4.2 we will consider the Schrödinger-like process

$$\begin{cases} iu_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \quad (3.1.20)$$

where the singularity is located on the boundary, and we will briefly discuss the well-posedness and controllability properties of (3.1.20).

The main results of this chapter have been announced in a short presentation in [37].

3.2. Pohozaev identity for A_λ

In this Section we rigorously justify the Pohozaev-type identity associated to A_λ. We discuss in detail the case C1. The details of the case C2 are let to the reader. In this latter case we only state the corresponding functional framework, see Subsection 3.2.2.

3.2.1. The case C1

In the following we introduce the functional framework which is used throughout the chapter and we discuss some of its properties.

Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a smooth domain which satisfies the case C1 and let $\lambda \leq \lambda(N)$, where $\lambda(N)$ was defined in (3.1.9). Thanks to inequality (3.1.7) the Hardy functional

$$B_\lambda[u] := \int_\Omega \Big(|\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \Big) dx, \quad (3.2.1)$$

is positive and finite for all $u \in C_0^\infty(\Omega)$. For any $\lambda \leq \lambda(N)$, $B_\lambda[u]$ induces a Hilbert space H_λ, defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||_{H_\lambda}^2 = B_\lambda[u], \quad \forall u \in C_0^\infty(\Omega). \quad (3.2.2)$$

We point out that the space H_λ was firstly analyzed by Vazquez and Zuazua [111] in the case of interior singularity. As emphasized above, it may be extended to the case of boundary singularity. In the subcritical case $\lambda < \lambda(N)$, it holds that $H_0^1(\Omega) = H_\lambda$, according to the estimates

$$\Big(1 - \frac{\max\{0, \lambda\}}{\lambda(N)}\Big) ||u||_{H_0^1(\Omega)} \leq ||u||_{H_\lambda}^2 \leq \Big(1 - \frac{\min\{0, \lambda\}}{\lambda(N)}\Big) ||u||_{H_0^1(\Omega)}^2, \quad \forall u \in C_0^\infty(\Omega),$$

which ensure the equivalence of the norms.

The critical space $H_{\lambda(N)}$ turns out to be slightly larger than $H_0^1(\Omega)$. We observe that $B_{\lambda(N)}[u]$ is finite for any $u \in H_0^1(\Omega)$, but it makes sense as an improper integral for more general distributions $u \in \mathcal{D}'(\Omega)$ i.e.

$$\exists \lim_{\varepsilon\to 0} \int_{\Omega\setminus B_\varepsilon(0)} \Big(|\nabla u|^2 - \frac{N^2}{4}\frac{u^2}{|x|^2}\Big)dx < \infty.$$

As it happens in the case of an interior singularity (see [110]), in general the meaning of $||u||_{H_{\lambda(N)}}$ does not coincide with the improper integral of $B_{\lambda(N)}[u]$. Following some ideas in [110], in the sequel we build a counterexample even in the case when the singularity is located on the boundary. We proceed by the absurd method so let us assume that

$$||u||^2_{H_{\lambda(N)}} = \lim_{\varepsilon\to 0} \int_{\Omega\setminus B_\varepsilon(0)} \Big(|\nabla u|^2 - \frac{N^2}{4}\frac{u^2}{|x|^2}\Big)dx. \tag{3.2.3}$$

Indeed, we consider Ω to be the unit ball in $\mathbb{R}^N$ centered at $(0,\dots,0,1)$ that is

$$\Omega := \big\{x \in \mathbb{R}^N_+ \ : \ |x'|^2 + (x_N-1)^2 \le 1\big\},$$

where x', x_N were defined in (3.1.3). Moreover, we introduce the distribution

$$e_1 = x_N|x|^{-N/2}J_0(z_{0,1}|x|),$$

where $z_{0,1}$ is the first positive zero of the Bessel function J_0. We observe that $B_{\lambda(N)}[e_1]$ is finite as an improper integral. Indeed, for the above defined Ω and e_1 we have

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon(0)} \Big(|\nabla e_1|^2 - \frac{N^2}{4}\frac{e_1^2}{|x|^2}\Big)dx &= \int_{\Omega\setminus B_\varepsilon(0)} |\nabla(J_0(z_{0,1}|x|))|^2 e_1^2 dx \\ &\quad + \int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|)x_N|x|^{-N/2}\nabla\big(x_N|x|^{-N/2}\big)\cdot\nu d\sigma, \end{aligned}\tag{3.2.4}$$

where $S_\varepsilon^{N-1,+} = \{x \in \mathbb{R}^N \mid |x| = \varepsilon, x_N > 0\}$ is the upper-half of the sphere $S_\varepsilon^{N-1} = \{x \in \mathbb{R}^N \mid |x| = \varepsilon\}$. Switching to polar coordinates in (3.2.4) and using basic properties of J_0 (in particular, $|J_0'(x)| \sim |x|/2$ as $x \to 0$) we get that

$$\int_{\Omega\setminus B_\varepsilon(0)} \Big(|\nabla e_1|^2 - \frac{N^2}{4}\frac{e_1^2}{|x|^2}\Big)dx = O(1), \text{ as } \varepsilon \to 0. \tag{3.2.5}$$

On the other hand, let us fix $\phi \in C_0^\infty(\Omega)$ and consider the transformation

$$e_1 - \phi = x_N|x|^{-N/2}w.$$

Then we have $w = J_0(z_{0,1}|x|) - |x|^{N/2}x_N^{-1}\phi$ and in particular $w \in C_0^\infty(\Omega)$ with $w(0) = J_0(0) > 0$. As before we have that

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon(0)} \Big(|\nabla(e_1-\phi)|^2 - \frac{N^2}{4}\frac{(e_1-\phi)^2}{|x|^2}\Big)dx &= \int_{\Omega\setminus B_\varepsilon(0)} |\nabla w|^2 (e_1-\phi)^2 dx \\ &\quad + \int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|)x_N|x|^{-N/2}\nabla\big(x_N|x|^{-N/2}\big)\cdot\nu d\sigma \end{aligned}\tag{3.2.6}$$

Applying polar coordinates (see e.g. [100], page 293) we obtain

$$\int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|)x_N|x|^{-N/2}\nabla\big(x_N|x|^{-N/2}\big)\cdot\nu d\sigma = \frac{N-2}{2}J_0^2(z_{0,1}\varepsilon)R, \tag{3.2.7}$$

where

$$R=\pi\Big(\int_0^\pi \cos^2\theta_1\sin^{N-2}\theta_1 d\theta_1\Big)\Big(\int_0^\pi \sin^{N-3}\theta_2 d\theta_2\Big)\dots\Big(\int_0^\pi \sin\theta_{N-2}d\theta_{N-2}\Big).$$

Therefore, due to (3.2.6)-(3.2.7) passing to the limit when $\varepsilon\to 0$ we obtain

$$||e_1-\phi||_{H_{\lambda(N)}}\geq \frac{N-2}{2}J_0^2(0)R>0, \quad \forall\phi\in C_0^\infty(\Omega),$$

provided $N>2$. This is in contradiction with the definition of $H_{\lambda(N)}$ which allows the existence of a sequence $\phi_n\in C_0^\infty(\Omega)$ converging to e_1 in $H_{\lambda(N)}$-norm ! Thererefore, the assumption of considering the definition of the $H_{\lambda(N)}$-norm as an improper integral of $B_{\lambda(N)}$ is false (at least for $N>2$). In other words, there are distributions $u\in H_{\lambda(N)}$ for which

$$||u||^2_{H_{\lambda(N)}}\neq \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\Big[|\nabla u|^2-\lambda(N)\frac{u^2}{|x|^2}\Big]dx. \tag{3.2.8}$$

Next we propose an equivalent norm on H_λ, $\lambda\leq\lambda(N)$, which overcomes the anomalous behavior in (3.2.8) and perfectly describes the meaning of the H_λ-norm.

The meaning of the H_λ-norm

For reasonable considerations that will be specified in (3.2.10), we introduce the functional

$$B_{\lambda,1}[u]=\int_\Omega\Big|\nabla u+\frac{N}{2}\frac{x}{|x|^2}u-\frac{e_N}{x_N}u\Big|^2dx+(\lambda(N)-\lambda)\int_\Omega\frac{u^2}{|x|^2}dx. \tag{3.2.9}$$

which is positive and finite for any $u\in C_0^\infty(\Omega)$ and $\lambda\leq\lambda(N)$. Next, we observe that, for any $\lambda\leq\lambda(N)$,

$$B_\lambda[u]=B_{\lambda,1}[u], \quad \forall u\in C_0^\infty(\Omega). \tag{3.2.10}$$

Besides, notice that both $B_{\lambda,1}[u]$ and $B_\lambda[u]$ are norms in H_λ and they coincide on $C_0^\infty(\Omega)$. Due to definition (3.2.2) of H_λ, we conclude that the H_λ could be defined as the closure of $C_0^\infty(\Omega)$ in the norm induced by $B_{\lambda,1}[u]$. Therefore, the H_λ-norm is characterized by the identification

$$||u||^2_{H_\lambda}=\lim_{\varepsilon\to 0}B^\varepsilon_{\lambda,1}[u], \quad \forall u\in H_\lambda, \tag{3.2.11}$$

where $\lambda\leq\lambda(N)$ and

$$B^\varepsilon_{\lambda,1}[u]:=\int_{|x|\geq\varepsilon}\Big|\nabla u+\frac{N}{2}\frac{x}{|x|^2}u-\frac{e_N}{x_N}u\Big|^2dx+(\lambda(N)-\lambda)\int_{|x|\geq\varepsilon}\frac{u^2}{|x|^2}dx, \quad \forall u\in H_\lambda.$$

Next in the chapter we will understand the meaning of the norm $||\cdot||_{H_\lambda}$ as in formula (3.2.11).

Main results

In what follows, $D(A_\lambda)$ stands for the domain of A_λ defined in (3.1.10). First of all, we note that standard elliptic estimates do not apply for A_λ if we want to obtain enough regularity for the normal derivative since the singularity $x = 0$ is located on the boundary. However, the following trace regularity result stated in Theorem 3.2.1 holds true.

Next, we claim the main results of Section 3.2.

Theorem 3.2.1 (Trace regularity)**.** *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded smooth domain satisfying the case C1. Let us consider $\lambda \leq \lambda(N)$ and $u \in D(A_\lambda)$. Then*

$$\Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2 \in L^1(\Gamma), \tag{3.2.12}$$

and moreover, there exists a positive constant $C = C(\Omega) > 0$ such that

$$\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2 d\sigma \leq C(||u||^2_{H_\lambda} + ||A_\lambda u||^2_{L^2(\Omega)}), \quad \forall u \in D(A_\lambda). \tag{3.2.13}$$

Theorem 3.2.2 (Pohozaev identity)**.** *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain satisfying the case C1 and let $\lambda \leq \lambda(N)$. If $u \in D(A_\lambda)$ we claim that*

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = -\int_\Omega A_\lambda u (x \cdot \nabla u) dx - \frac{N-2}{2}||u||^2_{H_\lambda}. \tag{3.2.14}$$

The proofs of Theorems 3.2.1, 3.2.2 are quite technical, so we need to apply some preliminary lemmas which are stated below. The proofs of Lemmas 3.2.1, 3.2.3 are postponed at the end of Subsection 3.2.1 while Lemma 3.2.2 is a consequence of an abstract approximation lemma in [2].

Lemma 3.2.1. *Supppose $u \in D(A_\lambda)$ and denote $f := A_\lambda u \in L^2(\Omega)$. Let us also consider $\theta_\varepsilon \in C_0^\infty(\Omega)$, $\varepsilon > 0$, a family of cut-off functions such that*

$$\theta_\varepsilon(x) = \theta_\varepsilon(|x|) = \begin{cases} 0, & |x| \leq \varepsilon \\ 1, & |x| \geq 2\varepsilon. \end{cases} \tag{3.2.15}$$

Assume $\vec{q} \in (C^2(\overline{\Omega}))^N$ is a vector field such that $\vec{q} = \nu$ on Γ, where ν denotes the outward normal to the boundary Γ (such an election of $\vec{q}$ can always be done in smooth domains, see [88], Lemma 3.1, page 29). Then we have the identity

$$\begin{aligned}
\frac{1}{2}\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2 \theta_\varepsilon d\sigma &= -\int_\Omega f(|x|^2 \vec{q}\cdot\nabla u \theta_\varepsilon) dx + 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx \\
&+ \sum_{i,j=1}^N \int_\Omega u_{x_i} u_{x_j} |x|^2 q^j_{x_i} \theta_\varepsilon dx - \int_\Omega |\nabla u|^2 (x\cdot\vec{q})\theta_\varepsilon dx \\
&- \frac{1}{2}\int_\Omega div\vec{q}|x|^2\Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)\theta_\varepsilon dx - \frac{1}{2}\int_\Omega |x|^2 \vec{q}\cdot\nabla\theta_\varepsilon\Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)dx \\
&+ \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx.
\end{aligned} \tag{3.2.16}$$

Lemma 3.2.2. *Assume $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^N$ verifying the case C1. For any $\varepsilon > 0$ aimed to be small, we consider the following approximation problem*

$$\begin{cases} A_{\lambda(N)-\varepsilon} u_\varepsilon = f, & x \in \Omega \\ u_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{3.2.17}$$

Then it holds

$$u_\varepsilon \to u \quad \text{strongly in } H_{\lambda(N)}, \text{ as } \varepsilon \to 0.$$

where u verifies the limit problem

$$-\Delta u - \lambda(N)\frac{u}{|x|^2} = f, \text{ in } \mathcal{D}'(\Omega).$$

Moreover

$$\varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \to 0, \text{ as } \varepsilon \to 0. \tag{3.2.18}$$

Lemma 3.2.3. *Assume Ω fulfills the case C1, let $\lambda \leq \lambda(N)$ and fix $f \in C^\infty(\Omega)$. Moreover, we assume that u_λ solves the problem*

$$\begin{cases} A_\lambda u_\lambda = f, & x \in \Omega, \\ u_\lambda \in H_\lambda. \end{cases} \tag{3.2.19}$$

Then u_λ satisfies the following upper bounds: there exists $r_0 < R_\Omega$ small enough and there exist constants $C_1, C_2 > 0$, independent of λ, such that

$$|u_\lambda(x)| \leq C_1 x_N |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left|\log \frac{1}{|x|}\right|^{1/2}, \quad a.e.\ x \in \Omega_{r_0}, \tag{3.2.20}$$

$$|\nabla u_\lambda(x)| \leq C_2 |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left|\log \frac{1}{|x|}\right|^{1/2}, \quad a.e.\ x \in \Omega_{r_0}, \tag{3.2.21}$$

where $\Omega_{r_0} := \Omega \cap B_{r_0}(0)$.

Notation: In order to facilitate the computations, in the sequel, we will write " $\gtrsim$ " and " $\lesssim$ " instead of " $\geq C$" respectively " $\leq C$" when we refer to universal constants C.

Proofs of Theorems 3.2.1, 3.2.2

Proof of Theorem 3.2.1. Following the proof of Theorem 3.1.1, as pointed out in Theorem 3.1.2 we are able to show that

$$\int_\Omega |x||\nabla u|^2 dx \lesssim ||u||^2_{H_\lambda}, \quad \forall u \in H_\lambda. \tag{3.2.22}$$

From the above estimate and the Cauchy-Schwartz inequality applied to identity (3.2.16) in Lemma 3.2.1 we obtain

$$\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 |x|^2 \theta_\varepsilon d\sigma \lesssim ||u||^2_{H_\lambda} + ||f||^2_{L^2(\Omega)}, \quad \forall u \in D(A_\lambda), \quad \forall \varepsilon > 0. \tag{3.2.23}$$

Combining the Fatou Lemma with (3.2.23) we finish the proof of Theorem 3.2.1.

□

Proof of Theorem 3.2.2. We split the proof in two main steps.

Step 1. The subcritical case

We recall that $H_\lambda = H_0^1(\Omega)$. Let $u \in D(A_\lambda)$ and put $f := A_\lambda u \in L^2(\Omega)$. By standard elliptic estimates we note that $u \in H^2(\Omega \setminus B_\varepsilon(0))$, for any $\varepsilon > 0$ small enough. Moreover, the normal derivative $\partial u/\partial\nu$ belongs to $L^2_{loc}(\partial\Omega \setminus \{0\})$. We multiply $A_\lambda u$ by $x \cdot \nabla u \theta_\varepsilon$, where θ_ε was defined in (3.2.15). After integration we get

$$\begin{aligned}\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2\theta_\varepsilon d\sigma &= -\int_\Omega f(x\cdot\nabla u)\theta_\varepsilon dx - \frac{N-2}{2}\int_\Omega \Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)\theta_\varepsilon dx\\ &\quad - \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)x\cdot\nabla\theta_\varepsilon dx + \int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx.\end{aligned} \tag{3.2.24}$$

Combining the Dominated Convergence Theorem (DCT) with Theorem 3.2.1 and condition (3.1.5), the left hand side of (3.2.24) converges i.e.

$$\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2\theta_\varepsilon d\sigma \to \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma, \qquad \text{as } \varepsilon \to 0.$$

In the right hand side of (3.2.24), we can directly pass to the limit term by term to obtain the identity (3.2.14) as follows. Firstly, since $x\cdot\nabla u \in L^2(\Omega)$ we have that

$$\begin{cases} |f(x\cdot\nabla u)\theta_\varepsilon| \le |f||x\cdot\nabla u| \in L^1(\Omega),\\ \theta_\varepsilon \to 1, \text{ a.e.}, \quad \text{as } \varepsilon \to 0,\end{cases}$$

and by DCT we obtain

$$\int_\Omega f(x\cdot\nabla u)\theta_\varepsilon dx \to \int_\Omega f(x\cdot\nabla u)dx, \qquad \text{as } \varepsilon \to 0.$$

Besides, from Hardy inequality and DCT we have

$$\int_\Omega |\nabla u|^2\theta_\varepsilon dx \to \int_\Omega |\nabla u|^2 dx, \quad \int_\Omega \frac{u^2}{|x|^2}\theta_\varepsilon dx \to \int_\Omega \frac{u^2}{|x|^2}dx, \qquad \text{as } \varepsilon \to 0.$$

Using the fact that $|\nabla\theta_\varepsilon| = O(1/\varepsilon)$ it follows that

$$\Big|\int_\Omega |\nabla u|^2 x\cdot\nabla\theta_\varepsilon dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} |\nabla u|^2 dx \to 0,$$

$$\Big|\int_\Omega \frac{u^2}{|x|^2} x\cdot\nabla\theta_\varepsilon dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} \frac{u^2}{|x|^2} dx \to 0,$$

$$\Big|\int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} |\nabla u|^2 dx \to 0,$$

as $\varepsilon \to 0$. With this we conclude the solvability of Theorem 3.2.2 in the subcritical case $\lambda < \lambda(N)$.

Step 2. The critical case $\lambda = \lambda(N)$

As before, let us consider $u \in D(A_{\lambda(N)})$ and define $f := A_{\lambda(N)} u \in L^2(\Omega)$. Our purpose is to show the validity of Theorem 3.2.2 for such u.

We proceed by approximations with subcritical values. More precisely, for $\varepsilon > 0$ small enough, we consider the problem

$$\begin{cases} A_{\lambda(N)-\varepsilon} u_\varepsilon = f, & x \in \Omega, \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \tag{3.2.25}$$

Applying Lemma 3.2.2 we obtain

$$u_\varepsilon \to u \text{ in } H_{\lambda(N)}, \quad \varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \to 0, \qquad \text{as } \varepsilon \to 0, \tag{3.2.26}$$

where u solves the limit problem. According to the Pohozaev identity applied to u_ε we get

$$\frac{1}{2} \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u_\varepsilon}{\partial \nu} \Big)^2 d\sigma = - \int_\Omega f (x \cdot \nabla u_\varepsilon) dx - \frac{N-2}{2} \Big(||u_\varepsilon||^2_{H_{\lambda(N)}} + \varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \Big). \tag{3.2.27}$$

Due to Theorem 3.1.1, the fact that $u_\varepsilon \to u$ in $H_{\lambda(N)}$ implies

$$x \cdot \nabla u_\varepsilon \to x \cdot \nabla u \text{ in } L^2(\Omega), \quad \text{as } \varepsilon \to 0.$$

Therefore, the right hand side in (3.2.27) converges to

$$H(u) := - \int_\Omega f (x \cdot \nabla u) dx - \frac{N-2}{2} ||u||^2_{H_{\lambda(N)}},$$

and therefore

$$\text{there exists } \lim_{\varepsilon \to 0} \frac{1}{2} \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u_\varepsilon}{\partial \nu} \Big)^2 d\sigma = H(u).$$

On the other hand, by standard elliptic regularity one can show that

$$\frac{\partial u_\varepsilon}{\partial \nu} \to \frac{\partial u}{\partial \nu} \text{ in } L^2_{\text{loc}}(\Gamma \setminus \{0\}) \text{ and } \frac{\partial u_\varepsilon}{\partial \nu} \to \frac{\partial u}{\partial \nu} \text{ a.e. on } \Gamma. \tag{3.2.28}$$

In the sequel, we discuss two different situations for the geometry of Ω.

Case 1. Assume Ω is flat in a neighborhood of zero (i.e. $x \cdot \nu = 0$). Then, as a consequence of DCT and (3.2.28) we note that

$$\lim_{\varepsilon \to 0} \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u_\varepsilon}{\partial \nu} \Big)^2 d\sigma = \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u}{\partial \nu} \Big)^2 d\sigma.$$

In consequence, u satisfies the Pohozaev identity, by passing to the limit in (3.2.27).

Case 2. We assume Ω is not necessarily flat at origin. We distinguish two cases when discussing the smoothness of f.

The case $f \in C^\infty(\Omega)$.

Next we apply Lemma 3.2.3 for u_ε the solution of problem (3.2.25). and we obtain

$$\Big|(x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2\Big| \le \Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2|x|^2 \le g, \text{ a.e. on } \Gamma,$$

where $g = |x|^{2-N}\Big|\log\frac{1}{|x|}\Big| \in L^1(\Gamma)$. Applying DCT we conclude

$$\lim_{\varepsilon\to 0}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2 d\sigma = \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma.$$

The case $f \in L^2(\Omega)$.

We consider $\{f_k\}_{k\ge1} \in C^\infty(\Omega)$ such that $f_k \to f$ in $L^2(\Omega)$, as $k\to\infty$.

Let us call u_k the solution of $A_{\lambda(N)}u_k = f_k$, for all $k \ge 1$. From the previous case, u_k satisfies

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_k}{\partial\nu}\Big)^2 d\sigma = -\int_\Omega f_k(x\cdot\nabla u_k)dx - \frac{N-2}{2}||u_k||^2_{H_{\lambda(N)}}. \tag{3.2.29}$$

We know that f_k is a Cauchy sequence in $L^2(\Omega)$, and due to

$$||u_k - u_l||_{H_{\lambda(N)}} \lesssim ||f_k - f_l||_{L^2(\Omega)} \to 0, \text{ as } k,l\to\infty,$$

we deduce that $\{u_k\}_{k\ge1}$ is Cauchy in $H_{\lambda(N)}$. Hence $u_k \to u$ in $H_{\lambda(N)}$ and

$$x\cdot\nabla u_k \to x\cdot\nabla u \text{ in } L^2(\Omega).$$

As a consequence we can pass to the limit in the right hand side of (3.2.29). In order to finish the proof, we need to pass to the limit in the left hand side. Indeed, in view of Theorem 3.2.1 we have

$$\int_\Gamma\Big(\frac{\partial(u_k-u_l)}{\partial\nu}\Big)^2|x|^2 d\sigma \lesssim ||u_k-u_l||^2_{H_\lambda} + ||f_k-f_l||^2_{L^2(\Omega)}.$$

Therefore $g_k := \frac{\partial u_k}{\partial\nu}|x|$ is a Cauchy sequence in $L^2(\Gamma)$ and g_k converges to, say, $g := \frac{\partial u}{\partial\nu}|x|$ in $L^2(\Gamma)$, as k goes to infinity. This suffices to say that

$$\lim_{k\to\infty}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_k}{\partial\nu}\Big)^2 d\sigma = \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma.$$

Therefore we conclude the proof of Theorem 3.2.2. □

Proofs of useful lemmas

Proof of Lemma 3.2.1. By standard elliptic estimates, we remark that $u \in H^2_{\mathrm{loc}}(\Omega\setminus\{0\})$. Thanks to that, when multiplying by $|x|\vec{q}\cdot\nabla u\theta_\varepsilon$ we are allowed to integrate by parts on Ω. Firstly, we obtain

$$\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = \int_\Gamma \frac{\partial u}{\partial\nu}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)d\sigma - \int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx.$$

Let us now compute the boundary term above. Since u vanishes on Γ it follows that

$$\nabla u = \frac{\partial u}{\partial \nu}\nu, \quad \text{on } \Gamma, \tag{3.2.30}$$

and moreover, $\vec{q} = \nu$ on Γ. Thanks to these we obtain

$$\int_\Gamma \frac{\partial u}{\partial \nu}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)d\sigma = \int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2|x|^2\theta_\varepsilon d\sigma.$$

Therefore,

$$\begin{aligned}\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = {} & \int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2|x|^2\theta_\varepsilon d\sigma - \int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u)\theta_\varepsilon dx \\ & - \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx.\end{aligned}$$

Let us compute the second term in the integration above. Doing various iterations we obtain

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u)\theta_\varepsilon dx = {} & 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx + \sum_{i,j=1}^N\int_\Omega u_{x_i}u_{x_j}|x|^2 q^j_{x_i}\theta_\varepsilon dx \\ & + \frac{1}{2}\sum_{i,j=1}^N\int_\Omega |x|^2 q^j(u^2_{x_i})_{x_j}\theta_\varepsilon dx\end{aligned} \tag{3.2.31}$$

For the last term in the integration above we get

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^N\int_\Omega |x|^2 q^j(u^2_{x_i})_{x_j}\theta_\varepsilon dx = {} & \frac{1}{2}\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2|x|^2\theta_\varepsilon d\sigma - \int_\Omega |\nabla u|^2(x\cdot\vec{q})\theta_\varepsilon dx \\ & - \frac{1}{2}\int_\Omega \mathrm{div}\vec{q}|x|^2|\nabla u|^2\theta_\varepsilon dx - \frac{1}{2}\int_\Omega |x|^2|\nabla u|^2\vec{q}\cdot\nabla\theta_\varepsilon dx.\end{aligned} \tag{3.2.32}$$

According to (3.2.31) and (3.2.32) we obtain

$$\begin{aligned}\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = {} & \frac{1}{2}\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2|x|^2\theta_\varepsilon d\sigma - 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx \\ & - \sum_{i,j=1}^N\int_\Omega u_{x_i}u_{x_j}|x|q^j_{x_i}\theta_\varepsilon dx + \int_\Omega |\nabla u|^2(x\cdot\vec{q})\theta_\varepsilon dx \\ & + \frac{1}{2}\int_\Omega \mathrm{div}\vec{q}|x|^2|\nabla u|^2\theta_\varepsilon dx + \frac{1}{2}\int_\Omega |x|^2|\nabla u|^2\vec{q}\cdot\nabla\theta_\varepsilon dx \\ & - \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx.\end{aligned} \tag{3.2.33}$$

On the other hand, it follows that

$$\int_\Omega \frac{u}{|x|^2}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = -\frac{1}{2}\int_\Omega \mathrm{div}\vec{q}u^2\theta_\varepsilon dx - \frac{1}{2}\int_\Omega \vec{q}\cdot\nabla\theta_\varepsilon u^2 dx. \tag{3.2.34}$$

From (3.2.33) and (3.2.34) we finally obtain the identity of Lemma 3.2.1. □

Proof of Lemma 3.2.3. For any $\lambda \leq \lambda(N)$ we fix $\phi_\lambda = x_N |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \Big| \log \frac{1}{|x|} \Big|^{1/2}$. Let us also consider the problem

$$\begin{cases} A_\lambda U_\lambda = |f|, & x \in \Omega, \\ U_\lambda \in H_\lambda. \end{cases} \tag{3.2.35}$$

The proof comprises several steps.

Step 1. Firstly let us check the validity of the Maximum Principle:

$$|u_\lambda(x)| \leq U_\lambda(x) \quad \text{a.e. in } \Omega. \tag{3.2.36}$$

Indeed, from the equations satisfied by U_λ, u_λ we obtain

$$-\Delta(U_\lambda \pm u_\lambda) - \lambda \frac{(U_\lambda \pm u_\lambda)}{|x|^2} = |f| \pm f \geq 0, \quad \forall x \in \Omega. \tag{3.2.37}$$

Multiplying (3.2.37) by the negative part $(U_\lambda \pm u_\lambda)^-$ we get the reverse Hardy inequality

$$\int_\Omega \Big[|\nabla (U_\lambda \pm u_\lambda)^-|^2 - \lambda \frac{[(U_\lambda \pm u_\lambda)^-]^2}{|x|^2} \Big] dx \leq 0. \tag{3.2.38}$$

From the non-attainability of the Hardy constant we necessary must have $(U_\lambda \pm u_\lambda)^- \equiv 0$ in Ω. Therefore, $U_\lambda \pm u_\lambda \geq 0$ in Ω, a fact which concludes (3.2.36).

Step 2. Next, we remark that there exists a positive constant $C_1 > 0$, independent of λ such that

$$-\Delta \phi_\lambda - \lambda \frac{\phi_\lambda}{|x|^2} \geq C_1, \quad \forall x \in \Omega.$$

Therefore, for some $C \geq ||f||_{L^\infty}/C_1$ we get

$$\begin{cases} -\Delta(C\phi_\lambda - U_\lambda) - \lambda \frac{(C\phi_\lambda - U_\lambda)}{|x|^2} \geq 0, & \forall x \in \Omega, \\ C\phi_\lambda - U_\lambda \geq 0, & x \in \Gamma. \end{cases} \tag{3.2.39}$$

Therefore, applying the Maximum Principle we obtain

$$U_\lambda \leq C\phi_\lambda, \quad \forall x \in \Omega, \quad \lambda \leq \lambda(N), \tag{3.2.40}$$

and the proof (3.2.20) is finished.

Step 3. For the estimate (3.2.21) we use a remark by Brezis-Marcus-Shafrir [24] as follows.

Fix $x \in \Omega_{r_0/2}$ and put $r = |x|/2$. We define then $\tilde{u}_\lambda(y) = u_\lambda(x + ry)$ where $y \in B_1(0)$. By direct computations we obtain

$$\begin{aligned} \Delta \tilde{u}_\lambda(y) &= r^2 \Delta u_\lambda(x + ry) = r^2 \Big(-f - \lambda \frac{u_\lambda(x+ry)}{|x+ry|^2} \Big) \\ &= -r^2 f - \lambda \frac{|x|^2}{4|x+ry|^2} \tilde{u}_\lambda(y). \end{aligned} \tag{3.2.41}$$

On the other hand, we remark that

$$\frac{4}{9} \leq \frac{|x|^2}{|x+ry|^2} \leq 4, \quad \forall y \in B_1(0).$$

By elliptic estimates it is easy to see that $\tilde{u}_\lambda \in C^1(B_1(0))$. Applying the interpolation inequality (see Evans [55]), we get that

$$\begin{aligned} |\nabla \tilde{u}_\lambda(0)| &\lesssim ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||\Delta \tilde{u}_\lambda||_{L^\infty(B_1(0))} \\ &\lesssim ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||f||_{L^\infty(\Omega)} \end{aligned} \tag{3.2.42}$$

Writing $\nabla \tilde{u}_\lambda$ in terms of ∇u_λ we obtain

$$|\nabla u_\lambda(x)| \lesssim \frac{1}{|x|}(||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||f||_{L^\infty(\Omega)}) \tag{3.2.43}$$

In addition, from (3.2.36) and (3.2.40) we have

$$\begin{aligned} ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} &= ||u_\lambda(x+ry)||_{L^\infty(B_1(0))} \\ &\lesssim \sup_{y \in B_1(0)} \left\{ (x_N + ry_N)|x+ry|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{1}{|x+ry|} \right|^{1/2} \right\} \\ &\lesssim x_N |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{1}{|x|} \right|^{1/2} + |x|^{-(N-2)/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{1}{|x|} \right|^{1/2} \\ &\lesssim |x|^{-(N-2)/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{1}{|x|} \right|^{1/2}, \end{aligned} \tag{3.2.44}$$

which is verified for all $x \in \Omega_{r_0}$, $y \in B_1(0)$. From (3.2.43) and (3.2.44) we obtain the estimate (3.2.21) which yields the proof of Lemma 3.2.3. □

3.2.2. Brief presentation of the case C2

Inequalities (3.1.7), (3.1.8) can be stated in a simplified form as follows.

Assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing the origin on the boundary. For any $\lambda \leq \lambda(N)$ and any $0 < \gamma < 2$ there exists a constant $C_1(\gamma, \Omega) \geq 0$ such that

$$\forall u \in H_0^1(\Omega), \qquad \int_\Omega \frac{u^2}{|x|^\gamma} dx + \lambda \int_\Omega \frac{u^2}{|x|^2} dx \leq \int_\Omega |\nabla u|^2 + C_1(\gamma, \Omega) \int_\Omega u^2 dx. \tag{3.2.45}$$

Functional framework via Hardy inequality

Let us now define the set

$$\mathcal{C}_\gamma := \left\{ C \geq 0 \ \text{ s. t. } \ \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 - \lambda(N) u^2/|x|^2 + Cu^2\right) dx}{\int_\Omega u^2/|x|^\gamma dx} \geq 1 \right\}. \tag{3.2.46}$$

Of course, $\mathcal{C}_\gamma$ is non empty due to inequality (4.1.4). Next we define

$$\mathcal{C}_0^\gamma = \inf_{C \in \mathcal{C}_\gamma} C. \tag{3.2.47}$$

Then, for any $\lambda \leq \lambda(N)$ we introduce the Hardy functional

$$B_\lambda[u] := \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \frac{u^2}{|x|^2} dx + \mathcal{C}_0^\gamma \int_\Omega u^2 dx, \tag{3.2.48}$$

which is positive for any $u \in H_0^1(\Omega)$ due to inequality (4.1.4) and the election of $\mathcal{C}_0^\gamma$. Then we define the corresponding Hilbert space H_λ as the closure of $C_0^\infty(\Omega)$ in the norm induced by $B_\lambda[u]$. Observe that for any $\lambda < \lambda(N)$ the identification $H_\lambda = H_0^1(\Omega)$ holds true. Indeed, if $\lambda < \lambda(N)$, we have

$$B_\lambda[u] \geq \Big(1 - \frac{\max\{0,\lambda\}}{\lambda(N)}\Big) \int_\Omega |\nabla u|^2 dx - \frac{\mathcal{C}_0^\gamma \max\{0,\lambda\}}{\lambda(N)} \int_\Omega u^2 dx. \tag{3.2.49}$$

On the other hand, from the definition of $\mathcal{C}_0^\gamma$ we obtain that there exists a constant $C_2 = C_2(\gamma) > 0$ such that

$$B_\lambda[u] \geq C_2 \int_\Omega u^2 dx. \tag{3.2.50}$$

Multiplying (4.1.16) by $\mathcal{C}_0^\gamma \max\{0,\lambda\}/(C_2\lambda(N))$ and summing to (4.1.15) we get that

$$B_\lambda[u] \geq C_\lambda \int_\Omega |\nabla u|^2 dx,$$

for some positive constant C_λ that converges to zero as λ tends to $\lambda(N)$.

Besides, in the critical case $\lambda = \lambda(N)$, H_λ is slightly larger than $H_0^1(\Omega)$. However, using cut-off arguments near the singularity (see e.g. [111]) we can show that

$$B_\lambda[u]_{\lambda(N)} \geq C_\varepsilon ||u||_{H^1(\Omega\setminus B_\varepsilon(0))}, \quad \forall u \in H_0^1(\Omega) \tag{3.2.51}$$

where C_ε is a constant going to zero as ε tends to zero.

Let us define the operator $A_\lambda := -\Delta - \lambda/|x|^2 + \mathcal{C}_0^\gamma I$ and define its domain as

$$D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}. \tag{3.2.52}$$

The norm of the operator A_λ is given by

$$||u||_{D(A_\lambda)} = ||u||_{L^2(\Omega)} + ||A_\lambda u||_{L^2(\Omega)}. \tag{3.2.53}$$

The meaning of the H_λ-norm

First of all we remark the validity of the identity

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega \frac{\Delta\Phi}{\Phi} u^2 dx = \int_\Omega \Big|\nabla u - \frac{\nabla\Phi}{\Phi} u\Big|^2 dx, \quad \forall u \in C_0^\infty(\Omega \setminus \{0\}), \tag{3.2.54}$$

which holds for any $\Phi \in C^1(\Omega \setminus \{0\})$ and $\Phi > 0$ in $\Omega \setminus \{0\}$. The proof of (3.2.54) applies direct integrations by parts.

Let us also consider $\phi(x) = \phi(|x|) \in C^\infty(\Omega)$ to be a cut-off function such that

$$\phi = \begin{cases} 1, & |x| \leq r_0/2, \ x \in \Omega \\ 0, & |x| \geq r_0, \ x \in \Omega, \end{cases} \tag{3.2.55}$$

where $r_0 > 0$ is meant to be small.

Case 1.Assume the points on the boundary Γ of Ω satisfy $x_N > 0$ in a neighborhood of the origin.

Next we consider $\Phi_1 = x_N|x|^{-N/2}$ which satisfies the equation

$$-\Delta\Phi_1 - \frac{N^2}{4}\frac{\Phi_1}{|x|^2} = 0, \quad \text{a. e. in } \Omega_{r_0}, \tag{3.2.56}$$

where $\Omega_{r_0} = \Omega \cap B_{r_0}(0)$ for some $r_0 > 0$ small enough. Applying (3.2.54) for $\phi = \phi_1$ from (3.2.56) we obtain

$$\int_{\Omega_{r_0}} |\nabla u|^2 dx - \frac{N^2}{4}\int_{\Omega_{r_0}} \frac{u^2}{|x|^2} dx = \int_{\Omega_{r_0}} \left|\nabla u - \frac{\nabla\Phi_1}{\Phi_1} u\right|^2 dx, \quad \forall u \in C_0^\infty(\Omega_{r_0}). \tag{3.2.57}$$

By a standard cut-off argument, due to (3.2.57) we remark that, there exist some weights $\rho_1, \rho_2 \in C^\infty(\Omega)$ depending on r_0, supported far from origin such that

$$\begin{aligned} B_\lambda[u] = &\int_\Omega \left|\nabla(u\phi) - \frac{\nabla\Phi_1}{\Phi_1}(u\phi)\right|^2 dx + \int_\Omega \rho_1|\nabla u|^2 dx \\ &+ (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in C_0^\infty(\Omega). \end{aligned} \tag{3.2.58}$$

Then the meaning of $||\cdot||_{H_\lambda}$-norm is characterized by

$$\begin{aligned} ||u||^2_{H_\lambda} = &\lim_{\varepsilon\to 0}\int_{x\in\Omega, |x|>\varepsilon} \left|\nabla(u\phi) - \frac{\nabla\Phi_1}{\Phi_1}(u\phi)\right|^2 dx + \int_\Omega \rho_1|\nabla u|^2 dx \\ &+ (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in H_\lambda, \quad \forall \lambda \le \lambda(N). \end{aligned} \tag{3.2.59}$$

Case 2. Assume the points on Γ satisfy $x_N \le 0$ in a neighborhood of the origin

In this case we consider $d = (x, \Gamma) = d(x)$ the function denoting the distance from a point $x \in \Omega$ to the boundary Γ. We remark that close enough to the origin the distribution

$$\Phi_2 = d(x)e^{(1-N)d(x)}|x|^{-N/2}\left|\log\frac{1}{|x|}\right|^{1/2}$$

satisfies

$$P := -\Delta\Phi_2 - \frac{N^2}{4|x|^2}\Phi_2 > 0, \quad \forall x \in \Omega_{r_0},$$

where $r_0 > 0$ is small enough. Due to this, there exist the weights $\rho_1, \rho_2 \in C^\infty(\Omega)$ depending on r_0 and supported away from origin, such that the meaning of the H_λ-norm is given by

$$\begin{aligned} ||u||^2_{H_\lambda} = &\lim_{\varepsilon\to 0}\int_{x\in\Omega, |x|>\varepsilon} \left|\nabla(u\phi) - \frac{\nabla\Phi_2}{\Phi_2}(u\phi)\right|^2 dx + (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx \\ &+ \int_\Omega \rho_1|\nabla u|^2 dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in H_\lambda, \quad \forall \lambda \le \lambda(N). \end{aligned} \tag{3.2.60}$$

Case 3. Assume that x_N changes sign on Γ at the origin.

This case can be analyzed through Case 2 above.

Then, the Pohozaev identity and related results presented in case C1 might be extended to case C2 by means of the weaker functional settings introduced above.

3.3. Applications to semi-linear equations

Pohozaev-type identities mostly apply to show non-existence results for non-linear elliptic problems. In particular, for applications to the semi-linear Laplace equation we refer mainly to [55], page 514.

In what follows we prove a non-existence result for a non-linear elliptic equation associated to A_λ, in the case of boundary singularity. In particular, the case $\lambda = 0$ in which no singularity occurs, corresponds to the standard case analyzed in [55]. To fix the ideas, let us assume $\lambda \leq \lambda(N)$ and consider $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a domain satisfying the case C1. In the sequel we use the notation

$$\alpha_\star := \frac{N+2}{N-2}$$

which stands for the critical Sobolev exponent.

Next we claim the main result

Theorem 3.3.1. *Let us consider the problem*

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = |u|^{\alpha-1} u, & x \in \Omega, \\ u = 0, & x \in \Gamma. \end{cases} \tag{3.3.1}$$

1. *Assume $\lambda \leq \lambda(N)$. If $1 < \alpha < \alpha_\star$ the problem* (3.3.1) *has non trivial solutions in H_λ. Moreover, if $1 < \alpha < \frac{N}{N-2}$ the problem* (3.3.1) *has non trivial solutions in $D(A_\lambda)$.*
2. *(non-existence). Assume $\lambda \leq \lambda(N)$ and let Ω be a smooth star-shaped domain (i.e. $x \cdot \nu \geq 0$, for all $x \in \Gamma$). If $\alpha \geq \alpha_\star$ the problem* (3.3.1) *does not have non trivial solutions in $D(A_\lambda)$.*

Proof of Theorem 3.3.1

Proof of item 1. The existence of non trivial solutions for (3.3.1) reduces to studying the minimization problem

$$I = \inf_{u \in H_\lambda, u \neq 0} \frac{||u||^2_{H_\lambda}}{||u||^{\alpha+1}_{L^{\alpha+1}(\Omega)}}.$$

Without losing generality, we may consider the normalization

$$I = \inf_{||u||_{L^{\alpha+1}(\Omega)}=1} J(u), \tag{3.3.2}$$

where $J : H_\lambda \to \mathbb{R}$ is defined by $J(u) = ||u||^2_{H_\lambda}$. Next we address the question of attainability of I in (3.3.2).

We note that J is continuous, convex, coercive in H_λ. Let $\{u_n\}_n$ be a minimizing sequence of I, i.e.,

$$J(u_n) \searrow I, \quad ||u_n||_{L^{\alpha+1}(\Omega)} = 1.$$

By the coercivity of J we have

$$||u_n||_{H_\lambda} \le C, \quad \forall n,$$

for some universal constant $C > 0$. Moreover, the embedding $H_\lambda \hookrightarrow L^{\alpha+1}(\Omega)$ is compact for any $\alpha < \alpha_\star$ (it can be deduced combining Theorem 3.1.2 and Sobolev inequality). Therefore,

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_\lambda, \\ u_n \to u & \text{strongly in } L^{\alpha+1}(\Omega). \end{cases} \tag{3.3.3}$$

According to (3.3.3) we get $||u||_{L^{\alpha+1}(\Omega)} = 1$. From the i.s.c. of the norm we have

$$I \le J(u) \le \liminf_{n\to\infty} J(u_n) = I,$$

and therefore $I = J(u)$ is attained by u, which, up to a constant, is a non-trivial solution of (3.3.1) in H_λ.

If $\alpha < N/(N-2)$ let us show that $u \in D(A_\lambda)$. Indeed, due to the compact embedding $H_\lambda \hookrightarrow L^q(\Omega)$, $q < 2N/(N-2)$, we have that $|u|^{\alpha-1}u \in L^2(\Omega)$. In consequence, $u \in D(A_\lambda)$. □

Proof of item 2. For the proof of non-existence we apply the Pohozaev identity in Theorem 3.2.2. In view of that we use the following lemma whose proof is postponed until the end of the section.

Lemma 3.3.1. *Assume $\lambda \le \lambda(N)$ and $1 < \alpha < \infty$. Then, any solution $u \in D(A_\lambda)$ of (3.3.1) satisfies the identity*

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = \Big(\frac{N}{1+\alpha} - \frac{N-2}{2}\Big)\int_\Omega |u|^{\alpha+1}dx. \tag{3.3.4}$$

The case $\alpha > \alpha_\star$.

Note that $x\cdot\nu \ge 0$ for all $x \in \Gamma$. Assuming $u \not\equiv 0$, from Lemma 3.3.1 we obtain $(N-2)/2 \le N/(\alpha+1)$ which is equivalent to $\alpha \le \alpha_\star$. This is in contradiction with the hypothesis on α. Therefore $u \equiv 0$ in Ω.

The case $\alpha = \alpha_\star$.

From Lemma 3.3.1, due to the criticality of α, u must satisfy

$$\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = 0.$$

Let us consider $\Omega = \{x \in \mathbb{R}^N_+ \mid |x'|^2 + (x_N - 1)^2 \le 1\}$ which is star-shaped. Therefore,

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{a.e. on } \Gamma.$$

Thus, the problem under consideration is reduced to the overdetermined system

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = |u|^{\frac{4}{N-2}} u, & x \in \Omega, \\ u = 0, & x \in \Gamma, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \Gamma. \end{cases} \tag{3.3.5}$$

Let us consider a compact subset $\Gamma' \subset \Gamma$ such that $x \cdot \nu > 0$ and $0 \notin \Gamma'$. Next, we extend Ω with a bounded set Ω_1 such that $\Omega_1 \cap \Omega = \emptyset$, $\partial\Omega_1 \cap \partial\Omega = \Gamma'$, $\tilde{\Omega} := \Omega \cup \Omega_1 \cup \Gamma'$.
For $\varepsilon > 0$ small enough we denote the sets $\Omega_\varepsilon := \Omega \setminus \{x \in \Omega \mid |x| < \varepsilon\}$, $\tilde{\Omega}_\varepsilon := \tilde{\Omega} \setminus \{x \in \Omega \mid |x| < \varepsilon\}$.

Next we consider the trivial prolongation of u to $\tilde{\Omega}$

$$\tilde{u} := \begin{cases} u, & x \in \Omega, \\ 0, & x \in \Omega_1. \end{cases} \tag{3.3.6}$$

The fact that $u \in D(A_\lambda)$ combined with the over-determined condition in (3.3.5), implies that $u \in H^2(\tilde{\Omega}_\varepsilon)$. Let us also show that $\tilde{u} \in H^2(\tilde{\Omega}_\varepsilon)$.

Indeed, thanks to (3.3.5) on Γ_0 we get that

$$\int_{\tilde{\Omega}_\varepsilon} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = -\int_{\tilde{\Omega}_\varepsilon} g\phi dx, \quad \forall \phi \in C_0^\infty(\tilde{\Omega}_\varepsilon), \tag{3.3.7}$$

where $g \in L^2(\tilde{\Omega}_\varepsilon)$ is given by

$$g = \begin{cases} \frac{\partial^2 u}{\partial x_i \partial x_j}, & x \in \Omega_\varepsilon, \\ 0, & x \in \Omega_1. \end{cases} \tag{3.3.8}$$

In particular we obtain that

$$\Delta \tilde{u} = \begin{cases} \Delta u, & x \in \Omega_\varepsilon, \\ 0 & x \in \Omega_1. \end{cases} \tag{3.3.9}$$

and $\tilde{u}$ verifies

$$-\Delta \tilde{u} - \frac{\lambda}{|x|^2} \tilde{u} = |\tilde{u}|^{\frac{4}{N-2}} \tilde{u} \ \text{ a.e. in } \tilde{\Omega}_\varepsilon \tag{3.3.10}$$

and $\tilde{u} \equiv 0$ in Ω_1. In other words we can write (3.3.10) as

$$-\Delta \tilde{u} = V(x) \tilde{u}, \quad x \in \tilde{\Omega}_\varepsilon,$$

where $V(x) := \frac{\lambda}{|x|^2} + |\tilde{u}|^{\frac{4}{N-2}}$. Note that $V \in L^\omega(\tilde{\Omega}_\varepsilon)$ for some $\omega > N/2$ and $\tilde{u}$ vanishes in Ω_1.

By this, we are in the hypothesis of the strong unique continuation result by Jerison and Kenig [81]. Therefore, $\tilde{u} \equiv 0$ in $\tilde{\Omega}_\varepsilon$ and in particular $u \equiv 0$ in Ω_ε, for any $\varepsilon > 0$. Hence, we conclude that $u \equiv 0$ in Ω. The proof of Theorem 3.3.1 is finished. □

Proof of Lemma 3.3.1. Since $u \in D(A_\lambda)$ we can apply the Pohozaev identity and we get

$$\frac{1}{2} \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u}{\partial \nu} \Big)^2 d\sigma = \int_\Omega -|u|^{\alpha-1} u (x \cdot \nabla u) dx - \frac{N-2}{2} ||u||^2_{H_\lambda}, \tag{3.3.11}$$

Next we show that

$$\int_{\Omega} |u|^{\alpha-1}u(x\cdot\nabla u)dx = -\frac{N}{1+\alpha}\int_{\Omega}|u|^{\alpha+1}dx. \tag{3.3.12}$$

We proceed by approximation arguments. For $\varepsilon > 0$ small enough we consider

$$I_\varepsilon := \int_{\Omega} |u|^{\alpha-1}u(x\cdot\nabla u)\theta_\varepsilon dx,$$

where θ_ε is a cut-off function supported in $\Omega\setminus B_\varepsilon(0)$. Due to the fact that $u\in H^2(\Omega\setminus\{0\})$ we can integrate by parts as follows:

$$\begin{aligned} I_\varepsilon &= -\frac{1}{2}\int_{\Omega}|u|^{\alpha-1}x\cdot\nabla(u^2)\theta_\varepsilon dx = \frac{1}{2}\int_{\Omega}u^2\mathrm{div}\big(|u|^{\alpha-1}x\theta_\varepsilon\big)dx \\ &= \frac{1}{2}\int_{\Omega}u^2\big(N|u|^{\alpha-1}\theta_\varepsilon + x\cdot\nabla\theta_\varepsilon|u|^{\alpha-1} + (\alpha-1)x\cdot\nabla u|u|^{\alpha-3}u\theta_\varepsilon\big)dx \\ &= \frac{N}{2}\int_{\Omega}|u|^{\alpha+1}\theta_\varepsilon dx + \frac{1}{2}\int_{\Omega}|u|^{\alpha+1}x\cdot\nabla\theta_\varepsilon dx - \frac{\alpha-1}{2}I_\varepsilon. \end{aligned} \tag{3.3.13}$$

Therefore we obtain

$$I_\varepsilon = \frac{N}{\alpha+1}\int_{\Omega}|u|^{\alpha+1}\theta_\varepsilon dx + \frac{1}{\alpha+1}\int_{\Omega}|u|^{\alpha+1}x\cdot\nabla\theta_\varepsilon dx. \tag{3.3.14}$$

From the equation itself it is easy to see that $|u|^{\alpha+1}\in L^1(\Omega)$ provided $u\in D(A_\lambda)$. Therefore, by the DCT we can pass to the limit as $\varepsilon\to 0$ in (3.3.14) to obtain the identity (3.3.12). On the other hand, multiplying (3.3.1) by u and integrating we obtain

$$||u||^2_{H_\lambda} = \int_{\Omega}|u|^{\alpha+1}dx,$$

Combining this with (3.3.12) and (3.3.11) we conclude the validity of (3.3.4). □

3.4. Applications to Controllability

In this section we study the controllability of the wave and Schrödinger equations with one singularity localized on the boundary of a smooth domain. Our motivation comes from the results shown in [109] in the context of an interior singularity.

For the sake of clarity, we will discuss in a detailed manner the case C1.

3.4.1. The wave equation. Case C1

In the sequel, we focus upon the controllability of the wave-like system

$$\begin{cases} u_{tt}-\Delta u-\lambda\frac{u}{|x|^2}=0, & (t,x)\in Q_T,\\ u(t,x)=h(t,x), & (t,x)\in(0,T)\times\Gamma_0,\\ u(t,x)=0, & (t,x)\in(0,T)\times(\Gamma\setminus\Gamma_0),\\ u(0,x)=u_0(x), & x\in\Omega,\\ u_t(0,x)=u_1(x), & x\in\Omega. \end{cases} \tag{3.4.1}$$

where $Q_T = (0,T) \times \Omega$ and Γ_0 is the boundary control region defined in (3.1.13) where the control $h \in L^2((0,T) \times \Gamma_0)$ is acting. We also assume $\lambda \leq \lambda(N)$. In view of the time-reversibility of the equation it is enough to consider the case where the target is

$$(\overline{u_0}, \overline{u_1}) = (0,0).$$

It is the so-called *null controllability problem.*

Well-posedness

Let us briefly discuss the well-posedness of system (3.4.1) in the corresponding functional setting.

Instead of (3.4.1) we firstly consider the more general system with non-homogeneous boundary conditions:

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in Q_T, \\ u(t,x) = g(t,x), & (t,x) \in \Sigma_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u_t(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{3.4.2}$$

where $\Sigma_T = (0,T) \times \Gamma$. The solution of (3.4.2) is defined by the transposition method (J. L. Lions [88]):

Definition 3.4.1. *Assume $\lambda \leq \lambda(N)$. For $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $g \in L^2((0,T) \times \Gamma)$, we say that u is a* **weak solution** *for (3.4.2) if*

$$\int_0^T \int_\Omega uf dxdt = -< u_0, z'(0) >_{L^2(\Omega)} + < u_1, z(0) >_{H'_\lambda, H_\lambda} - \int_0^T \int_\Gamma g \frac{\partial z}{\partial \nu} dxdt \quad \forall f \in \mathcal{D}(\Omega), \tag{3.4.3}$$

where $< \cdot, \cdot >$ represents the dual product between H_λ and its dual $H_\lambda^{'}$, and z is the solution of the non-homogeneous adjoint-backward problem

$$\begin{cases} z_{tt} - \Delta z - \lambda \frac{z}{|x|^2} = f, & (t,x) \in Q_T, \\ z(t,x) = 0, & (t,x) \in \Sigma_T, \\ z(T,x) = z'(T,x) = 0, & x \in \Omega. \end{cases} \tag{3.4.4}$$

Formally, (3.4.3) is obtained by multiplying the system (3.4.4) with u and integrating on Q_T. Using the Hardy inequalities above and the application of standard methods for evolution equations we obtain the following existence result.

Theorem 3.4.1 (well-posedness). *Assume that Ω satisfies C1. Let $T > 0$ be given and assume $\lambda \leq \lambda(N)$. For every $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and any $h \in L^2((0,T) \times \Gamma_0)$ there exists a unique weak solution of (3.1.12) such that*

$$u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H_\lambda^{'}). \tag{3.4.5}$$

Moreover, the solution of (3.1.12) satisfies

$$||(u, u_t)||_{L^\infty(0,T; L^2(\Omega) \times H'_\lambda)} \lesssim ||(u_0, u_1)||_{L^2(\Omega) \times H'_\lambda} + ||h||_{L^2((0,T) \times \Gamma_0)}. \tag{3.4.6}$$

The details of the proof of Theorem 3.4.1 are omitted since they follow the same steps as in [109].

Controllability and main results

It is by now classical that controllability of (3.4.1) is characterized through an observability inequality for the adjoint system as follows.

Given initial data $(u_0, u_1) \in L^2(\Omega) \times H_\lambda'$, a possible control $h \in L^2((0,T) \times \Gamma_0)$ must satisfy the identity

$$\int_0^T \int_{\Gamma_0} h \frac{\partial w}{\partial \nu} d\sigma dt - < u_t(0), w(0) >_{H_\lambda', H_\lambda} + < u(0), w_t(0) >_{L^2(\Omega)} = 0, \tag{3.4.7}$$

where w is the solution of the adjoint system

$$\begin{cases} w_{tt} - \Delta w - \lambda \frac{w}{|x|^2} = 0, & (t,x) \in Q_T, \\ w(t,x) = 0, & (t,x) \in \Sigma_T, \\ w(0,x) = w_0(x), & x \in \Omega, \\ w_t(0,x) = w_1(x), & x \in \Omega. \end{cases} \tag{3.4.8}$$

The operator $\mathcal{A}_\lambda$ defined by $\mathcal{A}_\lambda(w_0, w_1) = (w_1, \Delta w_0 + \lambda |x|^2 w_0)$ for all $(w_0, w_1) \in D(\mathcal{A}_\lambda) = D(A_\lambda) \times H_\lambda$, generates the wave semigroup i.e. $(\mathcal{A}_\lambda, D(\mathcal{A}_\lambda))$ is m-dissipative in $H_\lambda \times L^2(\Omega)$. In view of that, due to the theory of semigroups, the adjoint system is well-posed and more precisely it holds

Proposition 3.4.1 (see e.g. [109]). *(1) For any initial data $(w_0, w_1) \in H_\lambda \times L^2(\Omega)$ there exists a unique solution of (3.4.8)*

$$w \in C([0,T]; H_\lambda) \cap C^1([0,T]; L^2(\Omega)).$$

Moreover,

$$||(w, w_t)||_{L^\infty(0,T; H_\lambda \times L^2(\Omega))} \lesssim ||w_0||_{H_\lambda} + ||w_1||_{L^2(\Omega)} \tag{3.4.9}$$

(2) For any initial data $(w_0, w_1) \in D(A_\lambda) \times H_\lambda$ there exists a unique solution of (3.4.8) such that

$$w \in C([0,T]; D(A_\lambda)) \cap C^1([0,T]; H_\lambda) \cap C^2([0,T]; L^2(\Omega)).$$

Moreover

$$||(w, w_t)||_{L^\infty(0,T; D(A_\lambda) \times H_\lambda)} \lesssim ||w_0||_{D(A_\lambda)} + ||w_1||_{H_\lambda} \tag{3.4.10}$$

In the sequel, we claim some "hidden regularity" effect for the system (3.4.8) which may not be directly deduced from the semigroup regularity but from the equation itself.

Theorem 3.4.2 (Hidden regularity). *Assume $\lambda \leq \lambda(N)$ and w is the solution of* (3.4.8) *corresponding to the initial data $(w_0, w_1) \in H_\lambda \times L^2(\Omega)$. Then w satisfies*

$$\int_0^T \int_\Gamma (x \cdot \nu) \Big(\frac{\partial w}{\partial \nu} \Big)^2 d\sigma dt \lesssim \int_0^T \int_\Gamma \Big(\frac{\partial w}{\partial \nu} \Big)^2 |x|^2 d\sigma dt \lesssim ||w_0||_{H_\lambda}^2 + ||w_1||_{L^2(\Omega)}^2. \tag{3.4.11}$$

Moreover, w verifies the identity

$$\frac{1}{2} \int_0^T \int_\Gamma (x \cdot \nu) \Big(\frac{\partial w}{\partial \nu} \Big)^2 d\sigma dt = \frac{T}{2} (||w_0||_{H_\lambda}^2 + ||w_1||_{L^2(\Omega)}^2) + \int_\Omega w_t \Big(x \cdot \nabla w + \frac{N-1}{2} w \Big) \Big|_0^T dx. \tag{3.4.12}$$

Due to Theorem 3.4.2 the operator $(w_0, w_1) \mapsto \big(\int_0^T \int_{\Gamma_0} (x\cdot\nu)(\partial w/\partial\nu)^2 d\sigma dt\big)^{1/2}$ is a linear continuous map in $H_\lambda \times L^2(\Omega)$. Let $\mathcal{H}$ be the completion of this norm in $H_\lambda \times L^2(\Omega)$. We consider the functional $J : \mathcal{H} \to \mathbb{R}$ defined by

$$J(w_0, w_1)(w) := \frac{1}{2}\int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 d\sigma dt - < u_1, w_0 >_{H'_\lambda, H_\lambda} + (u_0, w_1)_{L^2(\Omega)}, \tag{3.4.13}$$

where w is the solution of (3.4.8) corresponding to initial data (w_0, w_1). Of course, $<\cdot,\cdot>_{H'_\lambda, H_\lambda}$ denotes the duality product. A control $h \in L^2((0,T)\times\Gamma_0)$ satisfying (3.4.7) could be chosen as $h = (x\cdot\nu)w_{\min}$ where $w_{\min}$ minimizes the functional J on $\mathcal{H}$ among the solutions w of (3.4.8) corresponding to the initial data $(u_0, u_1) \in H_\lambda^{'} \times L^2(\Omega)$. The existence of a minimizer of J is assured by the coercivity of J, which is equivalent to the *Observability inequality* for the adjoint system (3.4.8):

$$||w_0||^2_{H_\lambda} + ||w_1||^2_{L^2(\Omega)} \lesssim \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 d\sigma dt. \tag{3.4.14}$$

Conservation of energy.

For any $\lambda \le \lambda(N)$ and any fixed time $t \ge 0$, let us define the energy associated to (3.4.8):

$$E_w^\lambda(t) = \frac{1}{2}\big(||w_t(t)||^2_{L^2(\Omega)} + ||w(t)||^2_{H_\lambda}\big) \tag{3.4.15}$$

We note that adjoint system (3.4.8) is conservative and therefore

$$E_w^\lambda(t) = E_w^\lambda(0), \quad \forall\lambda \le \lambda(N), \quad \forall t \in [0,T]. \tag{3.4.16}$$

Next we claim our main results which answer to the controllability question.

Theorem 3.4.3 (Observability inequality). *For all $\lambda \le \lambda(N)$, there exists a positive constant $D_1 = D_1(\Omega,\lambda,T)$ such that for all $T \ge 2R_\Omega$ and any initial data $(w_0, w_1) \in H_\lambda \times L^2(\Omega)$, the solution of (3.4.8) verifies the observability inequality*

$$E_w^\lambda(0) \le D_1 \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 d\sigma dt. \tag{3.4.17}$$

The proof of Theorem 3.4.3 relies mainly on the method of multipliers (cf. [88]) and the so-called compactness-uniqueness argument (cf. [89]), combined with the new Hardy inequalities above. These results guarantee the exact controllability of (3.1.12) when the control acts on the boundary region Γ_0. In conclusion, we obtain

Theorem 3.4.4 (Controllability). *Assume that Ω satisfies the case C1 and $\lambda \le \lambda(N)$. For any time $T > 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$ there exists $h \in L^2((0,T)\times\Gamma_0)$ such that the solution of (3.4.1) satisfies*

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \textit{for all } x \in \Omega.$$

Proofs of main results

First of all, we need to justify that the solution w of adjoint system (3.4.8) possesses enough regularity to guarantee the integrability of the boundary term in (3.4.17). The justification is not trivial given the presence of the singularity at the boundary.

Proof of Theorem 3.4.2. We will proceed straightforward from Theorem 3.2.2.

Firstly, we consider initial data (w_0, w_1) in $D(\mathcal{A}_\lambda) = D(A_\lambda) \times H_\lambda$. Then, according to Proposition 3.4.1 we have

$$w \in C([0,T]; D(A_\lambda)) \cap C^1([0,T]; H_\lambda) \cap C^2([0,T]; L^2(\Omega)).$$

For a fixed time $t \in [0,T]$ we apply identity (3.2.16) in Lemma 3.2.1 with $f = -w_{tt}$. Passing to the limit when $\varepsilon \to 0$, by DCT and Fatou lemma we obtain $(\partial w/\partial\nu)|x| \in L^2(\Omega)$ and moreover

$$\begin{aligned}\frac{1}{2}\int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2 |x|^2 d\sigma &= \int_\Omega w_{tt}(|x|^2\vec{q}\cdot\nabla w)dx + 2\int_\Omega (x\cdot\nabla w)(\vec{q}\cdot\nabla w)dx \\ &+ \sum_{i,j=1}^N \int_\Omega w_{x_i}w_{x_j}|x|^2 q^j_{x_i}dx - \int_\Omega |\nabla w|^2(x\cdot\vec{q})dx \\ &- \frac{1}{2}\int_\Omega \mathrm{div}\vec{q}|x|^2\Big(|\nabla w|^2 - \lambda\frac{w^2}{|x|^2}\Big)dx. \end{aligned} \tag{3.4.18}$$

We have the following upper bounds for the terms in the right hand side of (3.4.18):

$$\Big|\int_\Omega |\nabla w|^2(x\cdot\vec{q})dx\Big|, \Big|\int_\Omega (x\cdot\nabla w)(\vec{q}\cdot\nabla w)dx\Big| \lesssim \int_\Omega |x||\nabla w|^2 dx \lesssim ||w||^2_{H_\lambda},$$

$$\Big|\sum_{i,j=1}^N \int_\Omega w_{x_i}w_{x_j}|x|^2 q^i_{x_i}dx\Big| \lesssim \int_\Omega |x|^2|\nabla w|^2 dx \lesssim ||w||^2_{H_\lambda},$$

$$\int_\Omega \mathrm{div}\vec{q}|x|^2\Big(|\nabla w|^2 - \lambda\frac{w^2}{|x|^2}\Big)dx \lesssim \int_\Omega |x|^2|\nabla w|^2 dx + \int_\Omega w^2 dx \lesssim ||w||^2_{H_\lambda},$$

which hold true due to Hardy inequality in Theorem 3.1.2. In consequence, integrating in time in (3.4.18) we obtain

$$\frac{1}{2}\int_0^T\int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2|x|^2 d\sigma dt \lesssim \Big|\iint_{Q_T} w_{tt}|x|^2\vec{q}\cdot\nabla w dxdt\Big| + \int_0^T ||w(t)||^2_{H_\lambda}dt. \tag{3.4.19}$$

Integrating by parts and applying the Cauchy-Schwartz inequality we have

$$\begin{aligned}\Big|\iint_{Q_T} w_{tt}|x|^2\vec{q}\cdot\nabla w dxdt\Big| &= \Big|\int_\Omega w_t|x|^2\vec{q}\cdot\nabla w\Big|_{t=0}^{t=T}dx - \iint_{Q_T} w_t|x|^2\vec{q}\cdot\nabla w_t dxdt\Big| \\ &\lesssim \int_\Omega \Big(w_t^2 + |x|^2|\nabla w|^2\Big)\Big|_{t=0}^{t=T}dx + \frac{1}{2}\iint_{Q_T} w_t^2\mathrm{div}(|x|^2\vec{q})dxdt\end{aligned} \tag{3.4.20}$$

From Theorem 3.1.1, the estimate (3.4.20) and due to the conservation of energy we obtain

$$
\begin{aligned}
\Big|\iint\limits_{Q_T} w_{tt}|x|^2\vec{q}\cdot\nabla w dxdt\Big| &\lesssim \int_\Omega (w_t^2(0,x)+w_t^2(T,x))dx + ||w(0)||^2_{H_\lambda} + ||w(T)||^2_{H_\lambda} + \iint\limits_{Q_T} w_t^2 dxdt \\
&\lesssim 2E_w^\lambda(0) + 2E_w^\lambda(T) + 2TE_w^\lambda(0) \\
&= (T+2)(||w_0||^2_{H_\lambda} + ||w_1||^2_{L^2(\Omega)}). \qquad (3.4.21)
\end{aligned}
$$

Since $x\cdot\nu \lesssim |x|^2$ on Γ, from (3.4.19) and (3.4.21) we conclude the inequality (3.4.11).

Next, we apply the Pohozaev identity for $w(t)$, $t\in[0,T]$. Indeed, integrating in time in Theorem 3.2.2 for $A_\lambda w = -w_{tt}$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Gamma (x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt = \\
&= \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T} dx - \iint\limits_{Q_T} w_t(x\cdot\nabla w_t)dxdt - \frac{N-2}{2}\int_0^T ||w(t)||^2_{H_\lambda}dt \\
&= \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T} dx + \frac{N}{2}\int_0^T ||w_t(t)||^2_{L^2(\Omega)}dt - \frac{N-2}{2}\int_0^T ||w(t)||^2_{H_\lambda}dt \\
&= \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T} dx + \frac{1}{2}\int_0^T \big(||w_t(t)||^2_{L^2(\Omega)} + ||w(t)||^2_{H_\lambda}\big)dt \\
&+ \frac{N-1}{2}\int_0^T \big(||w_t(t)||^2_{L^2(\Omega)} - ||w(t)||^2_{H_\lambda}\big)dt. \qquad (3.4.22)
\end{aligned}
$$

Multiplying the equation of (3.4.8) by w and integrating, the equipartition of the energy

$$
\int_\Omega ww_t\Big|_{t=0}^{t=T} dx = \int_0^T (||w_t(t)||^2_{L^2(\Omega)} - ||w(t)||^2_{H_\lambda})dt
$$

holds true. Due to the conservation of energy and from (3.4.22) we obtain precisely the identity (3.4.12). This yields the proof of Theorem 3.2.1 for initial data in the domain $D(\mathcal{A}_\lambda)$. Then, by density arguments, one can extend the results for less regular initial data $(w_0,w_1)\in H_\lambda\times L^2(\Omega)$. For such density arguments we refer to Lions [88], on pages 139-141. □

Proof of Theorem 3.4.3. In what follows we present the proof in the critical case $\lambda=\lambda(N)$, which is of main interest. The subcritical case $\lambda<\lambda(N)$ is let to the reader.

Step 1. Firstly, from Theorem 3.4.2 we remark that

$$
\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=0}^{t=T} dx + TE_w^{\lambda(N)}(0) \le \frac{1}{2}\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt. \qquad (3.4.23)
$$

For a fixed time $t = t_0 > 0$, by the Cauchy-Schwartz inequality we have

$$\begin{aligned}\Big|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0}dx\Big| &\le \frac{R_\Omega}{2}\int_\Omega w_t^2dx + \frac{1}{2R_\Omega}\int_\Omega\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)^2dx\\ &= \frac{R_\Omega}{2}||w_t||^2_{L^2(\Omega)} + \frac{1}{2R_\Omega}\bigg(\Big(\frac{N-1}{2}\Big)^2||w||^2_{L^2(\Omega)} +\\ &+ ||x\cdot\nabla w||^2_{L^2(\Omega)} + (N-1)\int_\Omega w(x\cdot\nabla w)dx\bigg),\end{aligned} \tag{3.4.24}$$

where R_Ω was defined in (3.1.6). On the other hand it follows

$$\int_\Omega w(x\cdot\nabla w)dx = \frac{1}{2}\int_\Omega x\cdot\nabla(w^2)dx = -\frac{1}{2}\int_\Omega \operatorname{div}(x)w^2dx = -\frac{N}{2}\int_\Omega w^2dx \tag{3.4.25}$$

Therefore from (3.4.24) and (3.4.25) we obtain

$$\begin{aligned}\Big|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0}dx\Big| &\le \frac{1}{2R_\Omega}||x\cdot\nabla w||^2_{L^2(\Omega)} + \frac{R_\Omega}{2}||w_t||^2_{L^2(\Omega)}\\ &- \frac{1}{2R_\Omega}\Big(\frac{N^2-1}{4}\Big)||w||^2_{L^2(\Omega)}\end{aligned}$$

Applying Theorem 3.1.1 we deduce

$$\Big|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0}dx\Big| \le R_\Omega E_w^{\lambda(N)}(t_0) - C||w(t_0)||^2_{L^2(\Omega)}, \tag{3.4.26}$$

for some constant C. Due to the conservation of energy, taking $t_0 = 0$ respectively $t_0 = T$ and summing in (3.4.26) we get

$$\Big|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=0}^{t=T}dx\Big| \le 2R_\Omega E_w^{\lambda(N)}(0) - C(||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)}), \tag{3.4.27}$$

From (3.4.23) and (3.4.27) we obtain

$$(T-2R_\Omega)E_w^{\lambda(N)}(0) \le \frac{1}{2}\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2d\sigma dt + C(||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)}). \tag{3.4.28}$$

Step 2. To get rid of the remaining term on the right hand side of (3.4.28) we need the following lemma.

Lemma 3.4.1. *There exists a positive constant $C = C(T,\Omega) > 0$ such that*

$$||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)} \le C\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2d\sigma dt \tag{3.4.29}$$

for all finite energy solutions of (3.4.8).

Combining Lemma 3.4.1 with (3.4.28), the Observability inequality is finally proved. □

Proof of Lemma 3.4.1. We apply a classical compactness-uniqueness argument. Suppose by contradiction that (3.4.29) does not hold. Then there exists a sequence (w_0^n, w_1^n) of initial data such that the corresponding solution w^n verifies

$$\frac{||w^n(0)||^2_{L^2(\Omega)} + ||w^n(T)||^2_{L^2(\Omega)}}{\int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial\nu}\Big)^2 d\sigma dt} \to \infty.$$

Normalizing we may suppose that

$$||w^n(0)||^2_{L^2(\Omega)} + ||w^n(T)||^2_{L^2(\Omega)} = 1, \qquad \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial\nu}\Big)^2 d\sigma dt \to 0. \tag{3.4.30}$$

From (3.4.28) and (3.4.30) we deduce that the sequence of energies $\{E_{w^n}^{\lambda(N)}(0)\}_n$ is uniformly bounded. In particular, we deduce that w^n is uniformly bounded in

$$C([0,T]; H_{\lambda(N)}) \cap C^1([0,T]; L^2(\Omega)).$$

Therefore, by extracting a subsequence

$$w^n \rightharpoonup w \text{ in } L^\infty(0,T;H_{\lambda(N)}) \text{ weakly-}\star, \text{ as } n\to\infty, \tag{3.4.31}$$

$$w_t^n \rightharpoonup w_t \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly-}\star, \text{ as } n\to\infty. \tag{3.4.32}$$

From Theorem 3.4.2 we obtain

$$\frac{\partial w^n}{\partial\nu}\sqrt{x\cdot\nu} \rightharpoonup \frac{\partial w}{\partial\nu}\sqrt{x\cdot\nu} \text{ in } L^\infty(0,T;L^2(\Gamma_0)) \text{ weakly-}\star, \text{ as } n\to\infty.$$

Furthermore, by lower semicontinuity and (3.4.30) we have

$$0 \le \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt \le \liminf_{n\to\infty} \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial\nu}\Big)^2 d\sigma dt = 0.$$

Hence

$$\int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt = 0,$$

and

$$(x\cdot\nu)\frac{\partial w}{\partial\nu} = 0, \qquad \text{a.e. on } \Gamma_0, \quad \forall t\in[0,T]. \tag{3.4.33}$$

On the other hand, from compactness and (3.4.31) we deduce that

$$w^n \to w \text{ in } L^\infty(0,T;L^2(\Omega)),$$

which combined with (3.4.30) yields

$$||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)} = 1. \tag{3.4.34}$$

To end the proof of Lemma 3.4.1 it suffices to observe that (3.4.33)-(3.4.34) lead to a contradiction. Indeed, in view of (3.4.33) and by Holmgreen's unique continuation we deduce that $w \equiv 0$ in Ω which is in contradiction with (3.4.34). □

Remark 3.4.1. *Unique continuation results may be applied far from the origin where the coefficient of the lower order term of the operator $-\partial_{tt} - \Delta - \lambda/|x|^2$ is analytic in time (actually, it is independent of time and bounded in space). The principal part coincides with the D'Alambertian operator, then one can apply Homlgreen's unique continuation to get $w = 0$ a.e. in $\Omega \setminus B(0,\varepsilon)$ for any $\varepsilon > 0$. In consequence, we will have $w \equiv 0$ in Ω, see e.g. [103].*

3.4.2. The Schrödinger equation

In this section we consider the Schrödinger-like equation

$$\begin{cases} iu_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in Q_T, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3.4.35}$$

Moreover, we assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain satisfying case C1 and $\lambda \leq \lambda(N)$. For the Schrödinger equation we define the Hilbert spaces $L^2(\Omega;\mathbb{C})$ and $H_0^1(\Omega;\mathbb{C})$ endowed with the inner products

$$< u, v >_{L^2(\Omega;\mathbb{C})} := \mathrm{Re} \int_\Omega u(x)\overline{v(x)}dx, \quad \forall u, v \in L^2(\Omega;\mathbb{C}),$$

$$< u, v >_{H_0^1(\Omega;\mathbb{C})} := \mathrm{Re} \int_\Omega \nabla u(x) \cdot \nabla\overline{v(x)}dx, \quad \forall u, v \in H_0^1(\Omega;\mathbb{C}).$$

For all $\lambda \leq \lambda(N)$, we also define the Hilbert space $H_\lambda(\Omega;\mathbb{C})$ as the completion of $H_0^1(\Omega;\mathbb{C})$ with respect to the norm associated with the inner product

$$< u, v >_{H_\lambda(\Omega;\mathbb{C})} := \mathrm{Re} \int_\Omega \Big(\nabla u(x) \cdot \nabla\overline{v(x)} - \lambda \frac{u(x)\overline{v(x)}}{|x|^2}\Big)dx, \quad \forall u, v \in H_0^1(\Omega;\mathbb{C}). \tag{3.4.36}$$

The spaces $L^2(\Omega;\mathbb{C})$, $H_0^1(\Omega;\mathbb{C})$, $H_\lambda(\Omega;\mathbb{C})$ inherit the properties of the corresponding real spaces. In order to simplify the notations, we will write $L^2(\Omega), H_0^1(\Omega), H_\lambda$ without making confusions.

As shown for the wave equation, the system (3.4.35) is well posed.

Theorem 3.4.5 (see [109]). *Let $T > 0$ be given and assume $\lambda \leq \lambda(N)$. For every $u_0 \in H_\lambda'$ and any $h \in L^2((0,T) \times \Gamma_0)$ the system* (3.4.35) *is well-posed, i.e. there exists a unique weak solution such that*

$$u \in C([0,T]; H_\lambda').$$

Moreover, there exists constant $C > 0$ such that the solution of (3.4.35) *satisfies*

$$||u||_{L^\infty(0,T;H_\lambda')} \leq C(||u_0||_{H_\lambda'} + ||h||_{L^2((0,T)\times\Gamma_0)}).$$

The system (3.4.35) is also controllable. More precisely, the control result states as follows.

Theorem 3.4.6. *The system* (3.4.35) *is controllable for any $\lambda \leq \lambda(N)$. More precisely, for any time $T > 0$, $u_0 \in H_\lambda'$ and $\overline{u_0} \in H_\lambda'$ there exists $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of* (3.4.35) *satisfies*

$$u(T,x) = \overline{u_0}(x) \quad \textit{for all } x \in \Omega.$$

As discussed in Subsection 3.4.1, the controllability is equivalent to the Observability inequality for the solution of the adjoint system

$$\begin{cases} iw_t + \Delta w + \lambda \frac{w}{|x|^2} = 0, & (t,x) \in Q_T, \\ w(t,x) = 0, & (t,x) \in (0,T) \times \Gamma, \\ w(0,x) = w_0(x), & x \in \Omega, \end{cases} \tag{3.4.37}$$

More precisely, if w solves (3.4.37), then for any time $T > 0$, there exists a positive constant C_T such that

$$||w_0||^2_{H_\lambda} \leq C_T \int_0^T \int_{\Gamma_0} (x \cdot \nu)\Big|\frac{\partial w}{\partial \nu}\Big|^2 d\sigma dt. \tag{3.4.38}$$

Observability (3.4.38) might be deduced directly using the multiplier identity stated in Lemma 3.4.2. The proof is let to the reader since it follows the same steps as in [109].

Lemma 3.4.2. *Assume* $\lambda \leq \lambda(N)$ *and* w *is the solution of* (3.4.37) *corresponding to the initial data* $w_0 \in H_\lambda$. *Then*

$$\int_0^T \int_\Gamma \Big|\frac{\partial w}{\partial \nu}\Big|^2 |x|^2 d\sigma dt \lesssim ||w_0||^2_{H_\lambda} \tag{3.4.39}$$

and w *satisfies the identity*

$$\frac{1}{2}\int_0^T \int_\Gamma (x \cdot \nu)\Big|\frac{\partial w}{\partial \nu}\Big|^2 d\sigma dt = T||w||^2_{H_\lambda} + \frac{1}{2} Im \int_\Omega w x \cdot \nabla \overline{w} dx\Big|_{t=0}^{t=T} dx.$$

Remark 3.4.2. *Besides, the proof of* (3.4.38) *can be deduced from the result valid for the wave equation. Indeed, the general theory presented in an abstract form in [106], assures the observability of systems like* $\dot{z} = iA_0 z$ *using results available for systems of the form* $\ddot{z} = -A_0 z$.

3.5. Appendix: sharp gradient bounds

Proof of Theorem 3.1.1. Without losing generality it is enough to consider two types of geometries for Ω as follows.

G1: The points on Γ satisfy $x_N \geq 0$ in the neighborhood of the origin.

G2; The points on Γ satisfy $x_N \leq 0$ in the neighborhood of the origin.

In the other intermediate case (when x_N changes sign at the origin) the result valid for case G2 still holds true since we can prove it for test functions extended from zero up to a domain satisfying G2.

The proof comprises several steps.

Step 1. Firstly we show that Theorem 3.1.1 is true in a neighborhood of $x = 0$. More precisely, there exists $r_0 = r_0(\Omega, N) > 0$ small enough, and $C = C(r_0)$ such that

$$\int_{\Omega_{r_0}} |x|^2 |\nabla w|^2 dx \leq R^2_\Omega \Big(\int_{\Omega_{r_0}} |\nabla w|^2 dx - \frac{N^2}{4}\int_{\Omega_{r_0}} \frac{w^2}{|x|^2} dx\Big) + C(r_0) \int_{\Omega_{r_0}} w^2 dx, \tag{3.5.1}$$

holds true for any function $w \in C_0^\infty(\Omega_{r_0})$, where $\Omega_{r_0} = \Omega \cap B_{r_0}(0)$.

Next we check the validity of Step 1. In view of that, let us consider a function ϕ which satisfies

$$-\Delta \phi \geq \frac{N^2}{4}\frac{\phi}{|x|^2}, \quad \phi > 0, \quad \forall x \in \Omega_{r_0}, \tag{3.5.2}$$

for some positive constant r_0. Such a function exists for each one of the cases G1-G2. Indeed, for the case G1 we may consider

$$\phi = x_N|x|^{-N/2} \tag{3.5.3}$$

and for case G2 we can take

$$\phi = d(x)e^{(1-N)d(x)}\Big|\log\frac{1}{|x|}\Big|^{1/2}|x|^{-N/2}. \tag{3.5.4}$$

With the transformation $w = \phi u$ for such ϕ as in (3.5.2) we get

$$|\nabla w|^2 = |\nabla\phi|^2u^2 + \phi^2|\nabla u|^2 + 2\phi u\nabla\phi\cdot\nabla u. \tag{3.5.5}$$

Integrating we obtain

$$\int_{\Omega_{r_0}}|\nabla w|^2dx = \int_{\Omega_{r_0}}|\nabla u|^2\phi^2dx - \int_{\Omega_{r_0}}\frac{\Delta\phi}{\phi}w^2. \tag{3.5.6}$$

On the other hand, multiplying in (3.5.5) by $|x|^2$ and integrating we obtain

$$\int_{\Omega_{r_0}}|x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}}|x|^2|\nabla\phi|^2u^2dx + \int_{\Omega_{r_0}}|x|^2\phi^2|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_{r_0}}|x|^2\nabla(\phi^2)\cdot\nabla(u^2)dx \tag{3.5.7}$$

For the last term in (3.5.7) we deduce

$$\frac{1}{2}\int_{\Omega_{r_0}}|x|^2\nabla(\phi^2)\cdot\nabla(u^2)dx = -\int_{\Omega_{r_0}}2\frac{x\cdot\nabla\phi}{\phi}w^2dx - \int_{\Omega_{r_0}}|x|^2|\nabla\phi|^2u^2dx - \int_{\Omega_{r_0}}\frac{\Delta\phi}{\phi}|x|^2w^2dx. \tag{3.5.8}$$

According to (3.5.7) and (3.5.8) we obtain

$$\int_{\Omega_{r_0}}|x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}}|x|^2\phi^2|\nabla u|^2dx - \int_{\Omega_{r_0}}\frac{2x\cdot\nabla\phi}{\phi}w^2dx - \int_{\Omega_{r_0}}\frac{\Delta\phi}{\phi}|x|^2w^2dx. \tag{3.5.9}$$

Taking into account the election of ϕ in (3.5.2) we have

$$-\frac{\Delta\phi}{\phi} = \frac{N^2}{4|x|^2} + P, \tag{3.5.10}$$

where $P \geq 0$ for any $x \in \Omega_{r_0}$. Then from (3.5.6) and (3.5.10) we have

$$\begin{aligned}\int_{\Omega_{r_0}}|x|^2\phi^2|\nabla u|^2dx \leq R_\Omega^2\int_{\Omega_{r_0}}\phi^2|\nabla u|^2 &= R_\Omega^2\Big(\int_{\Omega_{r_0}}|\nabla w|^2dx + \int_{\Omega_{r_0}}\frac{\Delta\phi}{\phi}w^2\Big)\\ &= R_\Omega^2\int_{\Omega_{r_0}}\Big(|\nabla w|^2 - \frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx - R_\Omega^2\int_{\Omega_{r_0}}Pw^2dx.\end{aligned} \tag{3.5.11}$$

From above and (3.5.9) it follows that

$$\begin{aligned}\int_{\Omega_{r_0}}|x|^2|\nabla w|^2dx \leq R_\Omega^2\int_{\Omega_{r_0}}\Big(|\nabla w|^2-\frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx - R_\Omega^2\int_{\Omega_{r_0}}Pw^2dx\\ -2\int_{\Omega_{r_0}}\frac{x\cdot\nabla\phi}{\phi}w^2dx+\int_{\Omega_{r_0}}\Big(\frac{N^2}{4|x|^2}+P\Big)|x|^2w^2dx\\ = R_\Omega^2\int_{\Omega_{r_0}}\Big(|\nabla w|^2-\frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx+\int_{\Omega_{r_0}}(|x|^2-R_\Omega^2)Pw^2dx\\ -2\int_{\Omega_{r_0}}\frac{x\cdot\nabla\phi}{\phi}w^2dx+\frac{N^2}{4}\int_{\Omega_{r_0}}w^2dx.\end{aligned} \tag{3.5.12}$$

In the case G1 (ϕ satisfies (3.5.3)) for r_0 small enough we have $P=0$ and

$$\Big|\frac{x\cdot\nabla\phi}{\phi}\Big|\leq C, \quad \forall x\in\Omega_{r_0},$$

holds for some positive constant C. Thanks to (3.5.12) we conclude the proof of Step 1 in the case G1. In the case G2 (ϕ satisfies (3.5.4)), for r_0 small enough we have

$$P>0, \quad \nabla d\cdot x\geq 0, \quad \forall x\in\Omega_{r_0}$$

Then, we remark

$$\frac{x\cdot\nabla\phi}{\phi}=\frac{x\cdot\nabla d}{d}+O(1),$$

and from above we finish the proof of Step 1 in this latter case.

Step 2. This step consists in applying a cut-off argument to transfer the validity of inequality (3.5.1) from Ω_{r_0} to Ω. More precisely, we consider a cut-off function $\theta\in C_0^\infty(\Omega)$ such that

$$\theta(x)=\begin{cases}1, & |x|\leq r_0/2,\\ 0, & |x|\geq r_0.\end{cases} \tag{3.5.13}$$

Then we split $w\in C_0^\infty(\Omega)$ as follows

$$w=\theta w+(1-\theta)w:=w_1+w_2. \tag{3.5.14}$$

Next let us firstly prove the following lemma.

Lemma 3.5.1. *Let us consider a weight function $\rho: C^\infty(\overline{\Omega})\to\mathbb{R}$ which is bounded and non negative. There exists $C(\Omega,\rho)>0$ such that the following inequality holds*

$$\int_\Omega\rho(x)\nabla w_1\cdot\nabla w_2dx\geq -C(\Omega,\rho,r)\int_\Omega|w|^2dx. \tag{3.5.15}$$

Proof of Lemma 3.5.1. From the boundary conditions, integrating by parts we have

$$\begin{aligned}
\int_\Omega \rho\nabla w_1\cdot\nabla w_2 dx &= \int \rho\theta(1-\theta)|\nabla w|^2dx + \int_\Omega \rho w\nabla w\cdot\nabla\rho(1-2\theta)dx - \int \theta|\nabla\theta|^2|w|^2dx \\
&\geq \frac{1}{2}\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} \nabla(|w|^2)\cdot\nabla\theta(1-2\theta)\rho dx - ||\rho||_\infty||D\theta||_\infty^2\int_\Omega |w|^2dx \\
&= -\frac{1}{2}\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} \operatorname{div}((1-2\theta)\rho\nabla\theta)|w|^2dx - ||\rho||_\infty||D\theta||_\infty^2\int_\Omega |w|^2dx \\
&\geq -C(||\rho||_{W^{1,\infty}}, ||\theta||_{W^{2,\infty}})\int_\Omega |w|^2dx. \qquad (3.5.16)
\end{aligned}$$

□

Now we are able to finalize Step 2. Indeed, splitting w as before we get

$$\int_\Omega |x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}} |x|^2|\nabla w_1|^2dx + \int_{\Omega\setminus\Omega_{r_0/2}} |x|^2|\nabla w_2|^2dx + 2\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} |x|^2\nabla w_1\cdot\nabla w_2dx$$

Applying (3.5.1) to w_1 in (3.5.14) we obtain

$$\begin{aligned}
\int_\Omega |x|^2|\nabla w|^2dx \leq R_\Omega^2\Big(\int_\Omega|\nabla w|^2dx - \frac{N^2}{4}\int_\Omega\frac{w_1^2}{|x|^2}dx\Big) + C\int_\Omega w^2dx- \\
-\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} 2(R_\Omega^2-|x|^2)\nabla w_1\cdot\nabla w_2dx. \qquad (3.5.17)
\end{aligned}$$

Adding $\rho = 2(R_\Omega^2-|x|^2)$ in Lemma 3.5.1, from (3.5.17) we get

$$\int_\Omega |x|^2|\nabla w|^2dx \leq R_\Omega^2\Big(\int_\Omega|\nabla w|^2dx - \frac{N^2}{4}\int_{\Omega_{r_0}}\frac{w_1^2}{|x|^2}dx\Big) + C(\Omega, r_0)\int_\Omega w^2dx. \qquad (3.5.18)$$

On the other hand we have

$$\int_{\Omega_{r_0}}\frac{w_1^2}{|x|^2} \geq \int_\Omega\frac{w^2}{|x|^2}dx - C(r_0)\int_\Omega w^2dx. \qquad (3.5.19)$$

From (3.5.18) and (3.5.19) the conclusion of Theorem 3.1.1 yields choosing r_0 small enough, $r_0 \leq R_\Omega$. □

Chapter 4

Control of the heat equation with boundary singular potential

Abstract. This chapter is devoted to analyze control properties for the heat equation with singular potential $-\mu/|x|^2$ arising at the boundary of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$. This problem was firstly studied by Vancostenoble and Zuazua [108] and then generalized by Ervedoza [54] in the context of interior singularity. Roughly speaking, these results showed that for any value of parameters $\mu \leq \mu_* := (N-2)^2/4$, the corresponding parabolic system can be controlled to zero with the control distributed in any open subset of the domain. The critical value μ_* stands for the best constant in the Hardy inequality with interior singularity.

When considering the case of boundary singularity a better critical Hardy constant is obtained, namely $\mu(N) := N^2/4$.

In this article we extend the previous results in [108], [54], to the case of boundary singularity. More precisely, we show that for any $\mu \leq \mu(N)$, we can lead the system to zero state using a distributed control in any open subset, excepting some cases where Ω changes the convexity at origin, in which cases the control is distributed in a semi-annulus surrounding the singularity.

We emphasize that our results cannot be obtained straightforwardly from the previous works [108], [54].

4.1. Introduction

In this article we present some new results concerning the exact controllability of the heat equation with singular quadratic potential $-\mu/|x|^2$.

Evolution problems with the potential $-\mu/|x|^2$ have been intensively studied in the recent decades. Among them, we remind the pionerring work by Baras and Goldstein [9] in which they considered the corresponding heat-like equation with the singularity localized in the interior of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ (If $N = 1$ they deleted the origin so that $0 \in \partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω). They derived necessary and sufficient conditions for such systems to being well-posed. More precisely, they showed the well-posedness holds true whenever $\mu \leq (N-2)^2/4$, whereas if $\mu > (N-2)^2/4$ there is instantaneously blow-up for the solution. The critical value $(N-2)^2/4$ is the best constant in the corresponding Hardy inequality (see e.g. [79], [78]). Later on, the issue of singular or degenerated potentials has been also analyzed by the control community. The authors in [109] analyzed the control and stabilization properties of the corresponding wave equation. Then they showed in Vancostenoble and Zuazua [108] that the corresponding heat equation can be controlled by a distributed control which sourrounds the singularity. This result has been generalized in Ervedoza [54] where any geometrical constraint of the control region was removed. We also mention the paper [31] and references there in studying the control of parabolic equations degenerating at origin. Recently, the work [54] has been slightly improved in [107] when studying some applications to inverse problems. In all situations above the authors showed that the well-posedness, control and stabilization are very much related to the classical Hardy inequality in which the best constant is $(N-2)^2/4$.

In this chapter we consider the heat equation with the potential $-\mu/|x|^2$, where the singularity $x=0$ is located on the boundary $\partial\Omega$ of a smooth open set $\Omega \subset \mathbb{R}^N$, $N \geq 1$. This work is aimed to extend to the case of boundary singularity the paper [54].

We point out that our results cannot be deduced straightforwardly from the case of interior singularity and requires an independent analysis. Our main tools concern in the Carleman estimates which is the classical way to prove observability properties for parabolic systems. The major difficulty consists in finding suitable weight functions to develop efficient Carleman estimates. In our case, the weights in [54], [108], are not even allowed to recover the results in the range of parameters $\mu \leq (N-2)^2/4$ shown in the case of interior singularity. A proper modification of the weights in [54], will be done here.

Before entering into details, let us fix some ideas.

Let $N \geq 2$ and consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \partial\Omega$, and let $\omega \subset \Omega$ be a non-empty open set. Assume also that $T > 0$ is fixed. We are interested in the question of controllability of the following problem

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1.1}$$

where $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega \times (0,T))$ is a function supported in the control region ω.

The null-controllability problem reads as follows: Given any $u_0 \in L^2(\Omega)$, find a function $f \in L^2(\omega \times (0,T))$ such that the solution of (4.1.1) satisfies

$$u(x,T) = 0, \qquad x \in \Omega. \tag{4.1.2}$$

In order to discuss the well-posedness and null-controllability of (4.1.1) we need to establish the proper functional framework which corresponds to the problem. The crucial role of this issue is played by a new critical value of μ which determines the features of system (4.1.1). More precisely, when moving the singularity from interior to the boundary, the critical Hardy constant jumps from $(N-2)^2/4$ to the critical value

$$\mu(N) := \frac{N^2}{4}. \tag{4.1.3}$$

This is guaranteed by the new Hardy inequality with boundary singularities (see e.g. [38], [59], [57]) stated in a simplified form as follows. For any $\mu \leq N^2/4$ there exists a constant

$C_1(\gamma,\Omega)$, such that for any $0<\gamma<2$ the inequality

$$\int_\Omega \frac{u^2}{|x|^\gamma}\,\mathrm{dx} + \mu\int_\Omega \frac{u^2}{|x|^2}\,\mathrm{dx} \leq \int_\Omega |\nabla u|^2 + C_1(\gamma,\Omega)\int_\Omega u^2\,\mathrm{dx}, \tag{4.1.4}$$

holds for all $u \in H_0^1(\Omega)$.

Moreover, an improved Hardy inequality (cf. Theorem 3.1.2) states that for any $0<\gamma<2$ and any $\mu \leq \mu(N) = N^2/4$ there exist $C_1(\gamma,\Omega)$, $C_2(\gamma,\Omega)>0$ such that

$$C_1\int_\Omega u^2 dx + \int_\Omega \Big[|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\Big]dx \geq C_2\int_\Omega \Big[|x|^{2-\gamma}|\nabla u|^2 + \frac{u^2}{|x|^\gamma}\Big]dx, \quad \forall u \in H_0^1(\Omega). \tag{4.1.5}$$

This inequality will be used in the proof of the Carleman estimates.

The main results of this paper are very much related to the geometry of Ω at the origin. In view of that, we say that Ω satisfies the property (P) if

$$(P):\begin{cases}\text{There exists a neighborhood } \mathcal{V}_0 \text{ of the origin } x=0 \text{ such that}\\ \qquad x\cdot\vec{n}\leq 0, \quad \forall\, x\in\partial\Omega\cap\mathcal{V}_0,\\ \text{where } \vec{n} \text{ denotes the outward normal vector.}\end{cases} \tag{4.1.6}$$

Roughly speaking, a domain $\Omega \subset \mathbb{R}^2$ with the property (P) is either concave or flat in a small neighborhood of the origin, as shown in the cases C1 and C2 in Figure 4.1.

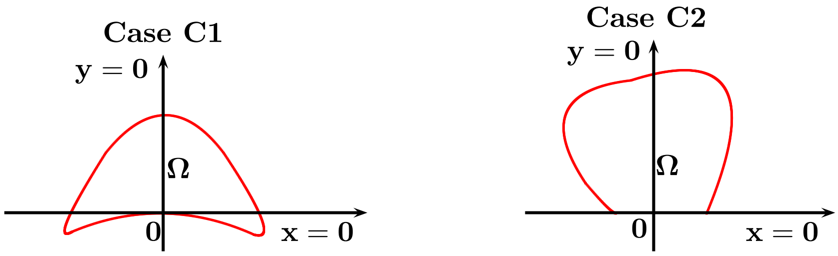

Figure 4.1: The geometry of $\Omega \subset \mathbb{R}^2$ is drawn in red color.

Otherwise, a domain $\Omega \subset \mathbb{R}^2$ not verifying the property (P) corresponds to Figure 4.2 below.

A domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ satisfying the property (P), is much more complicated to be drawn. However, in this multidimensional case, the property (P) may be described through the second fundamental form corresponding to $\partial\Omega$ in a neighborhood of origin (see e.g. [70]). Generally speaking, the property (P) is related to the non-convexity at origin. In particular, a domain strictly convex at origin does not satisfy the property (P).

Next, we are in conditions to state the main results of this paper.

Theorem 4.1.1 (Null-Controllability)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a smooth domain satisfying the property (P) and assume $\mu \leq \mu(N)$. Given any non-empty open set $\omega \subset \Omega$, for any time*

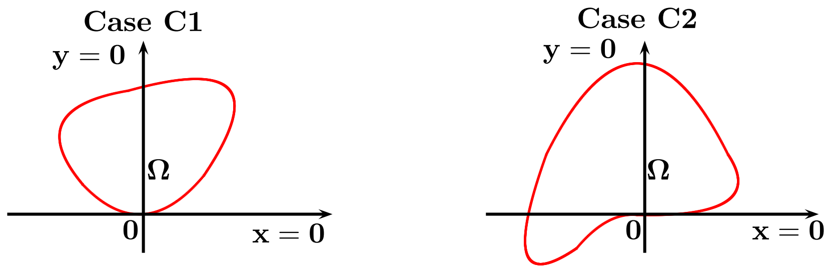

Figure 4.2: The geometry of $\Omega \subset \mathbb{R}^2$ is drawn in red color.

$T > 0$ and any initial data $u_0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega \times (0,T))$ such that the solution of (4.1.1) *satisfies* (4.1.2).

Theorem 4.1.2 (Null-Controllability). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a smooth domain which does not fulfill the property (P). We also assume $\mu \leq \mu(N)$ and let $r_1 > 0$ be a fixed constant.*

Then there exists $r_0 = r_0(\Omega) > 0$ (independent of r_1), $0 < r_0 < r_1$, such that for any time $T > 0$ and any initial data $u_0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega \times (0,T))$ distributed in the non-empty open set $\omega \subset \Omega$, defined by

$$\omega := \{x \in \Omega \mid r_0 < |x| < r_1\}, \tag{4.1.7}$$

such that the solution of (4.1.1) *satisfies* (4.1.2).

Following the by now classical HUM method (cf. [87]), the controllability property is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t w + \Delta w + \dfrac{\mu}{|x|^2} w = 0, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,T) = w_T(x), & x \in \Omega. \end{cases} \tag{4.1.8}$$

More precisely, when $\mu \leq \mu(N)$, we need to prove that there exists a constant C such that for all $w_T \in L^2(\Omega)$, the solution of (4.1.8) satisfies

$$\int_{\Omega} |w(x,0)|^2 \,\mathrm{dx} \leq C \iint_{\omega \times (0,T)} |w(x,t)|^2 \,\mathrm{dx\,dt}. \tag{4.1.9}$$

In order to prove (4.1.9), we will use a particular Carleman estimate , which is by now a classical technique in control theory.Indeed, the Carleman estimate we will derive later implies that for any solution w of (4.1.8),

$$\iint_{\Omega \times (\frac{T}{4}, \frac{3T}{4})} |w(x,t)|^2 \,\mathrm{dx\,dt} \leq C_T \iint_{\omega \times (0,T)} |w(x,t)|^2 \,\mathrm{dx\,dt}. \tag{4.1.10}$$

Let us show that (4.1.10) implies (4.1.9). Indeed, multiplying the system (4.1.8) by w and integrating in Ω we formally obtain

$$\frac{1}{2}\frac{d}{\mathrm{dt}}\int_\Omega w^2(x,t)\ \mathrm{dx} = \int_\Omega |\nabla w|^2\ \mathrm{dx} - \mu\int_\Omega \frac{w^2}{|x|^2}\ \mathrm{dx}.$$

From (4.1.4) we have that

$$\frac{1}{2}\frac{d}{\mathrm{dt}}\int_\Omega w^2(x,t)\ \mathrm{dx} \geq -C_1(\gamma,\Omega)\int_\Omega w^2(x,t)\ \mathrm{dx},$$

Then we get that the function $t \mapsto \exp^{2C_1(\gamma,\Omega)t}||w(\cdot,x)||^2_{L^2(\Omega)}$ is increasing. Then we obtain

$$\int_{T/4}^{3T/4}\int_\Omega w^2(x,t)\ \mathrm{dx\ dt} \geq \frac{T}{2}\exp^{-3TC_1(\gamma,\Omega)/2}\int_\Omega w^2(x,0)\ \mathrm{dx}, \tag{4.1.11}$$

From here and (4.1.10) we obtain (4.1.9).

Well-posedness via Hardy inequality

Let us now define the set

$$\mathcal{C}_\gamma := \Big\{ C \geq 0\ \text{ s. t. } \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega \big[|\nabla u|^2 - \mu(N)u^2/|x|^2 + Cu^2\big]dx}{\int_\Omega u^2/|x|^\gamma dx} \geq 1\Big\}. \tag{4.1.12}$$

Of course, $\mathcal{C}$ is non empty due to inequality (4.1.4). Next we define

$$\mathcal{C}_0^\gamma = \inf_{C\in\mathcal{C}_\gamma} C. \tag{4.1.13}$$

Then, for any $\mu \leq \mu(N) := N^2/4$ we introduce the Hardy functional

$$B_\mu[u] := \int_\Omega |\nabla u|^2 dx - \mu\int_\Omega \frac{u^2}{|x|^2}dx + \mathcal{C}_0^\gamma\int_\Omega u^2 dx, \tag{4.1.14}$$

which is positive for any test function due to inequality (4.1.4) and the election of $\mathcal{C}_0^\gamma$. Then we define the corresponding Hilbert space H_μ as the closure of $C_0^\infty(\Omega)$ in the norm induced by $B_\mu[\cdot]$. Observe that for any $\mu < \mu(N)$ the identification $H_\mu = H_0^1(\Omega)$ holds true. Indeed, if $\mu < \mu_N$ firstly we have

$$||u||^2_{H_\mu} \geq \big(1 - \frac{\mu}{\mu(N)}\big)\int_\Omega |\nabla u|^2 dx - \frac{\mathcal{C}_0^\gamma\mu}{\mu(N)}\int_\Omega u^2 dx. \tag{4.1.15}$$

On the other hand, from the definition of $\mathcal{C}_0$ we obtain that there is a constant $C > 0$ such that

$$||u||^2_{H_\mu} \geq C\int_\Omega u^2 dx. \tag{4.1.16}$$

Multiplying (4.1.16) by $\mathcal{C}_0\mu/(C_1\mu(N)$ and summing to (4.1.15) we get that

$$||u||^2_{H_\mu} \geq C_\mu\int_\Omega |\nabla u|^2 dx,$$

for some positive constant C_μ (C_μ tends to zero infinity as μ tends to $mu(N)$).

Besides, in the critical case $\mu = \mu_N$, H_μ is slightly larger than $H_0^1(\Omega)$. However, using cut-off arguments near the singularity (see e.g. [111]) we can show that

$$||u||_{H_{\mu(N)}} \geq C_\varepsilon ||u||_{H^1(\Omega \setminus \overline{B_\varepsilon(0)})}, \quad \forall u \in H_\mu, \tag{4.1.17}$$

where C_ε is a constant going to zero as ε tends to zero and $\overline{B_\varepsilon(0)}$ denotes the closure of the ball of radius ε centered at the origin.

Let us define now the operator $A_\mu := -\Delta - \mu/|x|^2 + \mathcal{C}_0^\gamma I$ together with its domain as

$$D(A_\mu) := \{u \in H_\mu \mid A_\mu u \in L^2(\Omega)\}. \tag{4.1.18}$$

The norm of this operator is given by

$$||u||_{D(A_\mu)} = ||u||_{L^2(\Omega)} + ||A_\mu u||_{L^2(\Omega)} \quad \forall \mu \leq \mu(N). \tag{4.1.19}$$

With these definitions, by standard semigroup-theory one can show that for any $\mu \leq \mu(N)$ the operator $(A_\mu, D(A_\mu))$ generates an analytic semigroup in the pivot space $L^2(\Omega)$ for the equation (4.1.1). For more details we refer to Theorem II.1, p. 3, [107], which can be adapted in the context of the space H_μ introduced above.

4.2. Null controllability in the case $\mu \leq N^2/4$

First of all, to simplify the presentation, we assume that $0 \notin \bar{\omega}$ otherwise it is straightforward since the control acts locally near the singularity. We also assume that $\Omega \cap \overline{B_1(0)}$ is included in Ω and $\overline{B_1(0)} \cap \bar{\omega}$ is empty. This can always be done by a scaling argument. In the sequel we also consider a nonempty subset $w_0 \subset\subset w$ whose role will be emphasized in the next paragraph.

In what follows we are going to justify the result of Theorem 4.1.1. For that we will apply Carleman estimates using a modified Fursikov-Imanuvilov [69] weight ψ which, in particular, satisfies

$$\nabla\psi \cdot x \geq 0, \quad \forall x \in \Omega \cap \mathcal{V}_0, \tag{4.2.1}$$

in a small neighborhood $\mathcal{V}_0$ of the origin.

In view of that, in the sequel we assume that the geometry of Ω satisfies the property (P). Without losing the generality, by a scaling argument we may assume that the property (P) reduces after normalization to

$$x \cdot \vec{n} \leq 0, \; \forall \; x \in \partial\Omega, \; |x| \leq 1, \tag{4.2.2}$$

4.2.1. Carleman estimates. Choice of the proper weight

As said in the introduction, the main tool we use to address the observability inequality (4.1.10) is a Carleman estimate.

The major problem when designing a Carleman estimate is the choice of a smooth weight function σ, which is in general assumed to be positive, and to blow up as t goes to zero and as t goes to T. Hence we are looking for a weight function σ that satisfies:

$$\begin{cases} \sigma(t,x) > 0, \quad (x,t) \in \Omega \times (0,T), \\ \lim\limits_{t\to 0^+} \sigma(t,x) = \lim\limits_{t\to T^-} \sigma(t,x) = +\infty, \quad x \in \Omega. \end{cases} \tag{4.2.3}$$

When shifting the singularity from interior up to the boundary the weight in Ervedoza [54] violates some necessary conditions to apply the Carleman estimates; the weight ψ blows up at origin and this violates the fact that ψ is constant on the boundary.

In view of that, we propose the weight

$$\sigma(t,x) = \theta(t)\Big(C_\lambda - |x|^2\psi - (\frac{|x|}{r_0})^\lambda \phi\Big), \quad \phi := e^{\lambda\psi(x)}, \tag{4.2.4}$$

where λ is a positive parameter aimed at being large and r_0 is a positive constant aimed to be small enough such that it verifies

$$r_0 \leq \min\Big\{\frac{1}{100}, \frac{1}{2-\gamma}\Big\} \min\Big\{\frac{1}{|D\psi|_{L^\infty(\Omega)} + |D^2\psi|_{L^\infty(\Omega)}}, \\ \Big(\frac{C_2}{8|D\psi|^2_{L^\infty(\Omega)} + 8|D^2\psi|^2_{L^\infty(\Omega)}}\Big)^{1/(\gamma-1)}, \Big(\frac{C_2}{2|\mu||D\psi|_{L^\infty(\Omega)}}\Big)^{1/(\gamma-1)}\Big\} \tag{4.2.5}$$

where γ is a fixed constant such that $1 < \gamma < 2$, C_2 stands for the constant in inequality (4.1.5), C_λ is big enough meant to assure the positivity of σ. Besides, ψ and θ are defined as follows. Firstly, we define θ by

$$\theta(t) = \Big(\frac{1}{t(T-t)}\Big)^k, \tag{4.2.6}$$

with $k = 1 + 2/\gamma$. Next we introduce ψ satisfying the conditions

$$\begin{cases} \psi > 1, & \forall\, x \in \Omega, \\ \psi \equiv 1, & \forall\, x \in \partial\Omega, \\ |\nabla\psi| \geq 2C_\Omega & \forall\, x \in \Omega \setminus \omega_0, \end{cases} \tag{4.2.7}$$

where the constant $C_\Omega > 0$ in (4.2.7) is such that verifies

$$|x \cdot \vec{n}| \leq C_\Omega |x|^2, \quad \forall x \in \partial\Omega. \tag{4.2.8}$$

Moreover, we assume that

$$x \cdot \nabla\psi \geq 0, \quad \forall\, x \in \Omega,\ |x| \leq 1. \tag{4.2.9}$$

Remark 4.2.1. *Such a function ψ satisfying conditions (4.2.7)-(4.2.9) exists. Indeed, let us assume $\tilde{\psi}$ is a smooth modification of by now the classical Fursikov-Imanuvilov weight (see [69]), with the following property close to the origin:*

$$\tilde{\psi} = d(x) > 0, \quad \forall\, x \in \Omega,\ |x| \leq 1, \tag{4.2.10}$$

where $d(x) = d(x, \partial\Omega)$ denotes the distance to the boundary. We remark that the distance d yields the conditions (4.2.7) since it satisfies the eikonal equation $|\nabla d(x)| = 1$ a.e. In particular, the assumptions (4.2.2)-(4.2.10) (see e.g. [8]) ensure the validity of (4.2.9).

To end up, we can choose $\psi := \delta(\tilde{\psi} + 1)$ which satisfies the constraints (4.2.7)-(4.2.8) for δ large enough.

Motivation for the choice of the weight σ.

Roughly speaking, in order to show the Carleman estimates below, the weight σ needs to fulfill some necessary conditions:

C1. $\nabla\sigma \cdot \vec{n} \geq 0$, for all $x \in \partial\Omega$.

C2. $-D^2\sigma(x)(\xi, \xi) > 0$, for all $\xi \in \mathbb{R}^N$, $x \in \Omega$, $|x| \leq r_0$, where r_0 is some positive constant independent of λ.

C3. $-\Delta\sigma > 0$, for all $|x| \geq r_0$, $x \notin \omega_0$, where r_0 is the same constant chosen in the condition C2.

C4. In order to get an expression in terms of the optimal Hardy inequality, the gradient term $\nabla\sigma$ must degenerate at origin at least as $|x|$.

The weight $\sigma = C_\lambda - |x|^2\psi - |x|^\lambda e^{\lambda\psi}$ fulfills the conditions C1, C2, C4. Moreover, it verifies the condition C3 provided

$$x \cdot \nabla\psi \geq 0, \quad \forall x \in \Omega \cap \mathcal{V}_0 \tag{4.2.11}$$

for some neighborhood $\mathcal{V}_0$ of the origin. The normalization by r_0 in (4.2.4) is required by technical reasons needed precisely in the proof of Lemma 4.2.4.

4.2.2. Main result

We claim that

Theorem 4.2.1. *There exists a positive constant K_1 and λ_0 such that for $\lambda \geq \lambda_0$ there exists $s_0(\lambda)$ such that for any $s \geq s_0$ we have*

$$\begin{aligned}
s\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\Big(\frac{|x|}{r_0}\Big)^\lambda \phi e^{-2s\sigma}|\nabla w|^2 \,\mathrm{dx\,dt} + s \iint\limits_{\Omega\times(0,T)} \theta e^{-2s\sigma}\Big[|x|^{2-\gamma}|\nabla w|^2 + \frac{w^2}{|x|^\gamma}\Big] \,\mathrm{dx\,dt} \\
+ s^3 \iint\limits_{\Omega\times(0,T)} \theta^3 e^{-2s\sigma}|x|^2|w|^2 \,\mathrm{dx\,dt} + s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} \theta^3 e^{-2s\sigma}\Big(\frac{|x|}{r_0}\Big)^{3\lambda}\phi^3|w|^2 \,\mathrm{dx\,dt} \\
\leq K\Big(s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\Big(\frac{|x|}{r_0}\Big)^\lambda \phi e^{-2s\sigma}|\nabla w|^2 \,\mathrm{dx\,dt} + s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3\Big(\frac{|x|}{r_0}\Big)^{3\lambda}\phi^3 e^{-2s\sigma}|w|^2 \,\mathrm{dx\,dt}\Big).
\end{aligned} \tag{4.2.12}$$

From Theorem 4.2.1 we can easily obtain the observability inequality (4.1.9) via Caccioppoli's inequality. The details could be reproduced step by step as in Section 2.2, page 12, [54].

4.2.3. Preliminaries and useful lemmas

Now, let us assume that w is a solution of (4.1.8) for some initial data $w_T \in H_0^1(\Omega)$, and define

$$z(t,x) = e^{-s\sigma(t,x)}w(t,x), \tag{4.2.13}$$

which obviously satisfies

$$z(T) = z(0) = 0 \qquad \text{in } H_0^1(\Omega) \tag{4.2.14}$$

due to the assumptions (4.2.3) on σ. The positive parameter s in (4.2.13) is aimed to be large. Then, plugging $w = z\exp(\sigma(t,x))$ in the equation (4.1.8), we obtain that z satisfies

$$\partial_t z + \Delta z + \frac{\mu}{|x|^2}z + 2s\nabla z \cdot \nabla\sigma + sz\Delta\sigma + z\Big(s\partial_t\sigma + s^2|\nabla\sigma|^2\Big) = 0, \quad (x,t) \in \Omega \times (0,T), \tag{4.2.15}$$

with the boundary condition

$$z = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{4.2.16}$$

Let us define a smooth positive radial function $\alpha(x) = \alpha(|x|)$ such that

$$\alpha(x) = \begin{cases} 0, & |x| \le r_0/2, \\ 1/N, & |x| \ge r_0, \end{cases} \tag{4.2.17}$$

where $r_0 > 0$ is selected as in (4.2.5). Setting

$$\begin{aligned} Sz &= \Delta z + \frac{\mu}{|x|^2}z + z\Big(s\partial_t\sigma + s^2|\nabla\sigma|^2\Big), \\ Az &= \partial_t z + 2s\nabla z \cdot \nabla\sigma + sz\Delta\sigma\Big(1+\alpha\Big), \\ Pz &= -s\alpha\Delta\sigma z \end{aligned} \tag{4.2.18}$$

One easily deduces from (4.2.15) that

$$Sz + Az + Pz = 0, \qquad ||Sz||^2 + ||Az||^2 + 2 < Sz, Az >= ||Pz||^2,$$

where $||\cdot||$ denotes the $L^2(\Omega \times (0,T))$ norm and $< \cdot,\cdot >$ the corresponding scalar product. Especially, the quantity

$$I =< Sz, Az > -\frac{1}{2}||s\alpha z\Delta\sigma||^2 \tag{4.2.19}$$

is non positive.

Lemma 4.2.1. *The following equality holds:*

$$
\begin{aligned}
I = -2s \iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z, \nabla z) \text{ dx dt} + s \iint\limits_{\partial\Omega\times(0,T)} |\partial_n z|^2 \, \partial_n \sigma \text{ ds dt} \\
- s \iint\limits_{\Omega\times(0,T)} |\nabla z|^2 \Delta\sigma \, \alpha \text{ dx dt} + \frac{s}{2} \iint\limits_{\Omega\times(0,T)} |z|^2 \Delta^2\sigma\Big(1+\alpha\Big) \text{ dx dt} \\
+ s \iint\limits_{\Omega\times(0,T)} |z|^2 \nabla\alpha\cdot\nabla\Delta\sigma \text{ dx dt} + \frac{s}{2} \iint\limits_{\Omega\times(0,T)} |z|^2 \Delta\sigma \, \Delta\alpha \text{ dx dt} \\
- \frac{1}{2} \iint\limits_{\Omega\times(0,T)} |z|^2 \Big(s\partial_{tt}^2\sigma + 2s^2\partial_t\Big(|\nabla\sigma|^2\Big)\Big) \text{ dx dt} - 2s^3 \iint\limits_{\Omega\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma,\nabla\sigma\Big) \text{ dx dt} \\
+ \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma\Big(s^2\partial_t\sigma + s^3|\nabla\sigma|^2\Big) \text{ dx dt} - \frac{s^2}{2} \iint\limits_{\Omega\times(0,T)} \alpha^2|z|^2|\Delta\sigma|^2 \text{ dx dt} \\
+ \mu s \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma \, \alpha \text{ dx dt} + 2\mu s \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4} x\cdot\nabla\sigma \text{ dx dt}
\end{aligned}
\tag{4.2.20}
$$

where $\partial_n = \vec{n}\cdot\nabla$ and ds *denotes the trace of the Lebesgue measure on $\partial\Omega$.*

Here we omit the proof of Lemma 4.2.1 since it may be found in [54]. It is worth mentioning that the upcoming computations justified by integrations by parts are done formally. However, we notice that the final estimates make sense in our functional framework. A priori the regularity of the operator $A_\mu := -\Delta - \mu/|x|^2 + \mathcal{C}_0 I$ is not enough to justify the integration by parts since the lack of regularity appears at the singular point $x = 0$. This issue is presented in a detailed manner in [37] in the context of the wave equation with singular potential localized on the boundary.

Now, we will decompose the term I in (4.2.20) into several terms that we handle separately.

Let us define I_{bd} the boundary term in identity (4.2.20):

$$
I_{bd} := \iint\limits_{\partial\Omega\times(0,T)} s|\partial_n z|^2 \, \partial_n\sigma \text{ ds dt.} \tag{4.2.21}
$$

Then define I_l as the sum of the integrals linear in σ which do not have any time derivative:

$$
\begin{aligned}
I_l = -2s \iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z,\nabla z) \text{ dx dt} - s \iint\limits_{\Omega\times(0,T)} |\nabla z|^2\Delta\sigma \, \alpha \text{ dx dt} + \frac{s}{2} \iint\limits_{\Omega\times(0,T)} |z|^2\Delta^2\sigma\Big(1+\alpha\Big) \text{ dx dt} \\
+ s \iint\limits_{\Omega\times(0,T)} |z|^2\nabla\alpha\cdot\nabla\Delta\sigma \text{ dx dt} + \frac{s}{2} \iint\limits_{\Omega\times(0,T)} |z|^2\Delta\sigma \, \Delta\alpha \text{ dx dt} \\
+ \mu s \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma \, \alpha \text{ dx dt} + 2\mu s \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4} x\cdot\nabla\sigma \text{ dx dt.}
\end{aligned}
\tag{4.2.22}
$$

We then consider the integrals involving non-linear terms in σ and without any time derivative, that is

$$I_{nl} = -2s^3 \iint\limits_{\Omega\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma, \nabla\sigma\Big) \text{ dx dt} + s^3 \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma|\nabla\sigma|^2 \text{ dx dt} \\ - \frac{s^2}{2} \iint\limits_{\Omega\times(0,T)} \alpha^2|z|^2|\Delta\sigma|^2 \text{ dx}. \quad (4.2.23)$$

We finally estimate the terms involving the time derivatives in σ:

$$I_t = -\frac{1}{2} \iint\limits_{\Omega\times(0,T)} |z|^2\Big(s\partial_{tt}^2\sigma + 2s^2\partial_t\Big(|\nabla\sigma|^2\Big)\Big) \text{ dx dt} + s^2 \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma\partial_t\sigma \text{ dx dt}. \quad (4.2.24)$$

The next step is to give convenient estimations for the terms defined above. In order to do that several lemmas are proved. Making the notations $\Omega_{r_0} = \Omega\cap B(0,r_0)$, $\mathcal{O} = \Omega\setminus(\omega_0\cup\Omega_{r_0})$, $\tilde{\mathcal{O}} = \Omega\setminus\Omega_{r_0}$ we successively obtain

Lemma 4.2.2. *It holds that $I_{bd} \geq 0$, for any $\lambda > 0$.*

Lemma 4.2.3. *There exists λ_0 such that for any $\lambda > \lambda_0$ and any $s > 0$ then*

$$I_l \geq C_2 s \iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big] \text{ dx dt} + C_7 s\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}+ \\ + C_5 s\lambda \iint\limits_{\Omega_{r_0}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-2}|\nabla z|^2 \text{ dx dt} - B_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \\ - C_6 s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}, \quad (4.2.25)$$

where C_2, C_7, C_5, C_6 and B_λ are constants uniform in s and λ, respectively uniform in s.

Lemma 4.2.4. *There exists λ_0 such that for any $\lambda > \lambda_0$ there exists $s_0(\lambda)$ such that for any $s > s_0$ it holds*

$$I_{nl} \geq \frac{s^3}{2} \iint\limits_{\Omega_{r_0}\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} + C_{15}s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} \theta^3\big(\frac{1}{r_0}\big)^{3\lambda}\phi^3|x|^{3\lambda}|z|^2 \text{ dx dt} \\ - C_{16}s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3\big(\frac{1}{r_0}\big)^{3\lambda}\phi^3|x|^{3\lambda}|z|^2 \text{ dx dt} \quad (4.2.26)$$

for some constants C_{15}, C_{16} uniform in s and λ.

Taking into account the negative terms in the expression of I_l that we want to get rid of, we define

$$I_r = I_t - B_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt}. \quad (4.2.27)$$

Then

Lemma 4.2.5. *There exists λ_0 such that for any $\lambda > \lambda_0$ there exists $s_0(\lambda)$ such that for any $s > s_0$ we have*

$$|I_r| \leq \frac{C_2 s}{2} \iint\limits_{Q_T} \theta \frac{|z|^2}{|x|^\gamma} \,\mathrm{dx\,dt} + \frac{C_{15}}{2} s^3\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3 \big(\frac{1}{r_0}\big)^{3\lambda} \phi^3 |x|^{3\lambda} |z|^2 \,\mathrm{dx\,dt} + \frac{s^3}{4} \iint\limits_{Q_T} \theta^3 |x|^2 |z|^2 \,\mathrm{dx\,dt}. \tag{4.2.28}$$

Carleman inequality in the variable z

From lemmas above we obtain

Theorem 4.2.2. *There exists a positive constant K and λ_0 such that for $\lambda \geq \lambda_0$ there exists $s_0(\lambda)$ such that for any $s \geq s_0$ we have*

$$\begin{aligned} s\lambda^2 &\iint\limits_{\mathcal{O}\times(0,T)} \theta \big(\frac{|x|}{r_0}\big)^\lambda \phi |\nabla z|^2 \,\mathrm{dx\,dt} + s \iint\limits_{\Omega\times(0,T)} \theta \Big[|x|^{2-\gamma} |\nabla z|^2 + \frac{z^2}{|x|^\gamma} \Big] \,\mathrm{dx\,dt} \\ &+ s^3 \iint\limits_{\Omega\times(0,T)} \theta^3 |x|^2 |z|^2 \,\mathrm{dx\,dt} + s^3\lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} \theta^3 \big(\frac{|x|}{r_0}\big)^{3\lambda} \phi^3 |z|^2 \,\mathrm{dx\,dt} \\ &\leq K \Big(s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta \big(\frac{|x|}{r_0}\big)^\lambda \phi |\nabla z|^2 \,\mathrm{dx\,dt} + s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3 |x|^{3\lambda} \phi^3 |z|^2 \,\mathrm{dx\,dt} \Big). \end{aligned} \tag{4.2.29}$$

Undoing the variables in Theorem 4.4.2 we obtain the conclusion of Theorem 4.2.1.

4.3. Proofs of technical lemmas

Lemma 4.3.1. *Let us consider the function*

$$\tau_\phi(x) = \Big(\frac{|x|}{r_0}\Big)^\lambda e^{\lambda\psi}.$$

The Hessian matrix

$$D^2\tau_\phi = \Big[\frac{\partial^2 \tau_\phi}{\partial x_i \partial x_j}\Big]_{i,j=1,\ldots N},$$

is positive definite uniformly in $|x| \leq r_0$, for any $\lambda \geq 2$. More precisely, for any $\lambda \geq 2$ and any $x \in \mathbb{R}^N$ with $|x| \leq r_0$ we have

$$\langle D^2\tau_\phi \xi, \xi\rangle \geq \frac{\lambda}{2}\Big(\frac{|x|}{r_0}\Big)^{\lambda-2} \phi |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \tag{4.3.1}$$

Proof of Lemma 4.3.1. Firstly, we write $D^2\tau_\phi(\xi,\xi) = \phi(1/r_0)^\lambda S_\phi$ where

$$\begin{aligned} S_\phi :&= \lambda|\xi|^2|x|^{\lambda-2} + \lambda(\lambda-2)|x\cdot\xi|^2|x|^{\lambda-4} + 2\lambda^2(x\cdot\xi)(\xi\cdot\nabla\psi)|x|^{\lambda-2} + \\ &+ \lambda|x|^\lambda D^2\psi(\xi,\xi) + \lambda^2|x|^\lambda|\nabla\psi\cdot\xi|^2. \end{aligned} \tag{4.3.2}$$

Next we use to the inequality

$$2\lambda^2(x\cdot\xi)(\nabla\psi\cdot\xi)|x|^{\lambda-2}\le a\lambda^2|x\cdot\xi|^2|x|^{\lambda-4}+\frac{\lambda^2}{a}|x|^{\lambda}|\nabla\psi\cdot\xi|^2,\quad \forall a>0.$$

Combining this with (4.3.2) we obtain

$$S_\phi\ge\lambda|\xi|^2|x|^{\lambda-2}+(\lambda^2-2\lambda-a\lambda^2)|x\cdot\xi|^2|x|^{\lambda-4}+\lambda|x|^{\lambda}D^2\psi(\xi,\xi)+\big(\lambda^2-\frac{\lambda^2}{a}\big)|x|^{\lambda}|\nabla\psi\cdot\xi|^2.$$

Next we choose $a>0$ such that $\lambda^2-2\lambda-a\lambda^2=0$, that is $a=(\lambda-2)/\lambda$. For this value of a we remark that

$$S_\phi\ge\lambda|\xi|^2|x|^{\lambda-2}+\lambda|x|^{\lambda}D^2\psi(\xi,\xi)-\frac{2\lambda^2}{\lambda-2}|x|^{\lambda}|\nabla\psi|^2|\xi|^2,\quad\forall x,\forall\xi. \tag{4.3.3}$$

Then we deduce that

$$S_\phi\ge\frac{\lambda}{2}|x|^{\lambda-2}|\xi|^2+\lambda|x|^{\lambda-2}|\xi|^2\big(\frac{1}{2}-\frac{2\lambda}{\lambda-2}|x|^2|D\psi|^2_\infty-|x|^2|D^2\psi|^2_\infty\big), \tag{4.3.4}$$

and finally

$$S_\phi\ge\frac{\lambda}{2}|x|^{\lambda-2}|\xi|^2$$

holds true for $\lambda\ge2$ and for any $|x|\le r_0$. This yields the proof of Lemma 4.3.1. □

Proof of Lemma 4.2.2. It suffices to prove that $\nabla\sigma\cdot\vec{n}\ge0$ in $\partial\Omega\times(0,T)$. Firstly we have

$$\nabla\sigma=\theta(t)\Big[-2x\psi-|x|^2\nabla\psi-\big(\frac{1}{r_0}\big)^{\lambda}(\lambda x|x|^{\lambda-2}+\lambda|x|^{\lambda}\nabla\psi)\phi\Big]. \tag{4.3.5}$$

Dept to the first two conditions in (4.2.7) we have

$$\nabla\psi\cdot\vec{n}=-|\nabla\psi|,\quad\forall x\in\partial\Omega.$$

In consequence we get

$$\nabla\sigma\cdot\vec{n}=\theta(t)\Big[-2x\cdot\vec{n}+|x|^2|\nabla\psi|+\lambda\big(\frac{1}{r_0}\big)^{\lambda}\phi|x|^{\lambda-2}(|x|^2|\nabla\psi|-x\cdot\vec{n})\Big].$$

Due to (4.2.8) we have

$$\nabla\sigma\cdot\vec{n}\ge\theta(t)\Big[|x|^2(|\nabla\psi|-2C_\Omega)+\lambda\phi\big(\frac{|x|}{r_0}\big)^{\lambda}(|\nabla\psi|-C_\Omega)\Big]$$

which is positive since ψ satisfies the third condition in (4.2.7). This completes the proof of Lemma 4.2.2. □

Proof of Lemma 4.2.3. The computations here require a more careful analysis. First of all, we make the notations

$$\sigma_{x^2}=-\theta(t)\tau_{x^2},\ \text{where}\ \tau_{x^2}=|x|^2\psi,$$

respectively

$$\sigma_\phi=-\theta(t)\tau_\phi,\ \text{where}\ \tau_\phi=\big(\frac{|x|}{r_0}\big)^{\lambda}\phi.$$

and

$$\tau = \tau_{x^2} + \tau_\phi.$$

Next we write some formulas for τ_{x^2} and τ_ϕ that we are going to use in our computations. More precisely, for all $x \in \mathbb{R}^N$ and any $i, j = 1, \ldots, N$ we have

$$\begin{aligned} \partial_{x_i}\tau_{x^2} &= 2x_i\psi + |x|^2\partial_{x_i}\psi \\ \partial^2_{x_ix_j}\tau_{x^2} &= 2\delta_{i,j}\psi + 2x_i\partial_{x_j}\psi + 2x_j\partial_{x_i}\psi + |x|^2\partial^2_{x_ix_j}\psi, \end{aligned} \tag{4.3.6}$$

respectively

$$\begin{aligned} \partial_{x_i}\tau_\phi &= \Big(\frac{1}{r_0}\Big)^\lambda\big(\lambda x_i|x|^{\lambda-2} + \lambda|x|^\lambda\partial_{x_i}\psi\big)\phi, \\ \partial^2_{x_ix_j}\tau_\phi &= \Big(\frac{1}{r_0}\Big)^\lambda\Big[\lambda\delta_{ij} + \lambda(\lambda-2)x_ix_j|x|^{\lambda-4} + \lambda^2 x_j\partial_{x_i}\psi|x|^{\lambda-2} + \lambda^2 x_i\partial_{x_j}\psi|x|^{\lambda-2} + \\ &\quad + \lambda|x|^\lambda\partial_{x_ix_j}\psi + \lambda^2|x|^\lambda\partial_{x_i}\psi\partial_{x_j}\psi\Big]\phi, \end{aligned} \tag{4.3.7}$$

Computations for I_l.

Next we split I_l in two parts as $I_l := I_l^1 + I_l^2$ where

$$\begin{aligned} I_l^1 &:= -2s\iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z, \nabla z)\ \mathrm{dx\ dt} - s\iint\limits_{\Omega\times(0,T)} \Delta\sigma\alpha|\nabla z|^2\ \mathrm{dx\ dt} + 2\mu s\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4}x\cdot\nabla\sigma\ \mathrm{dx\ dt}, \\ I_l^2 &:= \frac{s}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta^2\sigma\Big(1+\alpha\Big)\ \mathrm{dx\ dt} + s\iint\limits_{\Omega\times(0,T)} |z|^2\nabla\alpha\cdot\nabla\Delta\sigma\ \mathrm{dx\ dt} + \\ &\quad + \frac{s}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta\sigma\ \Delta\alpha\ \mathrm{dx\ dt} + \mu s\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma\ \alpha\ \mathrm{dx\ dt} \end{aligned} \tag{4.3.8}$$

Moreover, we split I_l^1 as $I_l^1 := I^1_{l,x^2} + I^1_{l,\phi}$ where

$$I^1_{l,x^2} = -2s\iint\limits_{\Omega\times(0,T)} D^2\sigma_{x^2}(\nabla z, \nabla z)\ \mathrm{dx\ dt} - s\iint\limits_{\Omega\times(0,T)} \Delta\sigma_{x^2}\alpha|\nabla z|^2\ \mathrm{dx\ dt} + 2\mu s\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4}x\cdot\nabla\sigma_{x^2}\ \mathrm{dx\ dt} \tag{4.3.9}$$

$$I^1_{l,\phi} := -2s\iint\limits_{\Omega\times(0,T)} D^2\sigma_\phi(\nabla z, \nabla z)\ \mathrm{dx\ dt} - s\iint\limits_{\Omega\times(0,T)} \Delta\sigma_\phi\alpha|\nabla z|^2\ \mathrm{dx\ dt} + 2\mu s\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4}x\cdot\nabla\sigma_\phi\ \mathrm{dx\ dt}. \tag{4.3.10}$$

Estimates for I^1_{l,x^2}:

Using the relations (4.3.6) we have

$$I^1_{l,x^2} = 4s \iint\limits_{\Omega\times(0,T)} \theta\Big[|\nabla z|^2 - \mu\frac{|z|^2}{|x|^2}\Big]\psi \text{ dx dt} + 8s \iint\limits_{\Omega\times(0,T)} \theta(x\cdot\nabla z)(\nabla\psi\cdot\nabla z) \text{ dx dt}$$
$$+2s \iint\limits_{\Omega\times(0,T)} \theta|x|^2 D^2\psi(\nabla z,\nabla z) \text{ dx dt} - 2\mu s \iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|^2}(\nabla\psi\cdot x) \text{ dx dt} - s \iint\limits_{\Omega\times(0,T)} \Delta\sigma_{x^2}\alpha|\nabla z|^2 \text{ dx dt}. \tag{4.3.11}$$

Next, we estimate the first term in I^1_{l,x^2} applying the Hardy inequality (4.1.5). We proceed as follows Firstly, fixing the time and integrating in the space variable we get the identities

$$\int_\Omega z\nabla z\cdot\nabla\psi \text{ dx} = -\frac{1}{2}\int_\Omega z^2\Delta\psi \text{ dx}, \tag{4.3.12}$$

$$\int_\Omega |x|^{2-\gamma} z\nabla z\cdot\nabla\psi \text{ dx} = -\frac{1}{2}\int_\Omega |x|^{2-\gamma}\Delta\psi z^2 \text{ dx} - \frac{(2-\gamma)}{2}\int_\Omega (x\cdot\nabla\psi)|x|^{-\gamma}z^2 dx \tag{4.3.13}$$

Secondly, we apply the Hardy inequality (4.1.5) setting $u := z\sqrt{\psi}$. Then, according to (4.3.12)-(4.3.13) and integrating in time we have

$$\begin{aligned}
C_1 &\iint\limits_{\Omega\times(0,T)} \theta\psi z^2 \text{ dx dt} + \iint\limits_{\Omega\times(0,T)} \theta\Big[|\nabla z|^2 - \mu\frac{z^2}{|x|^2}\Big]\psi \text{ dx dt} \\
&\qquad + \frac{1}{4}\iint\limits_{\Omega\times(0,T)} \theta z^2\frac{|\nabla\psi|^2}{\psi} \text{ dx dt} - \frac{1}{2}\iint\limits_{\Omega\times(0,T)} \theta z^2\Delta\psi \text{ dx dt} \\
&\geq C_2 \iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big]\psi \text{ dx dt} + \frac{C_2}{4}\iint\limits_{\Omega\times(0,T)} \theta\frac{|\nabla\psi|^2}{\psi}z^2 \text{ dx dt} \\
&\quad - \frac{C_2}{2}\iint\limits_{\Omega\times(0,T)} \theta|x|^{2-\gamma}\Delta\psi z^2 \text{ dx dt} - C_2\frac{(2-\gamma)}{2}\iint\limits_{\Omega\times(0,T)} \theta(x\cdot\nabla\psi)|x|^{-\gamma}z^2 \text{ dx dt}.
\end{aligned}$$

Since r_0 satisfies (4.2.5) we have $C_2\psi/(2|x|^\gamma) \geq C_2(2-\gamma)|D\psi|_\infty|x|^{1-\gamma}/2$ for $|x| \leq r_0$. Then we easily obtain

$$\begin{aligned}
\iint\limits_{\Omega\times(0,T)} \theta\Big[|\nabla z|^2 - \mu\frac{z^2}{|x|^2}\Big]\psi \text{ dx dt} &\geq \frac{C_2}{2}\iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big]\psi \text{ dx dt} \\
&- \Big(C_1|\psi|_\infty + \frac{|D\psi|^2_\infty}{4} + \frac{1}{2}|D^2\psi|_\infty + \frac{C_2}{2}R_\Omega^{2-\gamma}|D^2\psi|_\infty + \\
&\qquad + \frac{C_2}{2}(2-\gamma)r_0^{1-\gamma}|D\psi|_\infty\Big)\iint\limits_{\Omega\times(0,T)} \theta z^2 \text{ dx dt}.
\end{aligned} \tag{4.3.14}$$

Denoting $C_3 := C_1|\psi|_\infty + |D\psi|^2_\infty/4 + |D^2\psi|_\infty/2 + C_2R_\Omega^{2-\gamma}|D^2\psi|_\infty/2 + C_2(2-\gamma)r_0^{1-\gamma}|D\psi|_\infty/2$ and $C_2' := C_2/2$, where $R_\Omega = \sup_{x\in\overline{\Omega}}|x|$, we get

$$\iint\limits_{\Omega\times(0,T)} \theta\Big[|\nabla z|^2 - \mu\frac{z^2}{|x|^2}\Big]\psi \text{ dx dt} \geq C_2' \iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{|z|^2}{|x|^\gamma}\Big]\psi \text{ dx dt}$$
$$- C_3 \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt}. \quad (4.3.15)$$

From (4.3.15) we obtain

$$\begin{aligned} I^1_{l,x^2} \geq 4sC_2' &\iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big] \text{ dx dt} - 4sC_3 \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \\ &- 8s|D\psi|_\infty \iint\limits_{\Omega\times(0,T)} \theta|x||\nabla z|^2 \text{ dx dt} - 2s|D^2\psi|_\infty \iint\limits_{\Omega\times(0,T)} \theta|x|^2|\nabla z|^2 \text{ dx dt} \\ &- 2s|\mu||D\psi|_\infty \iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|} \text{ dx dt} - s \iint\limits_{\Omega\times(0,T)} \Delta\sigma_{x^2}\alpha|\nabla z|^2 \text{ dx dt}. \end{aligned} \quad (4.3.16)$$

Since r_0 satisfies (4.2.5) we observe that $2C_2'|x|^{2-\gamma} \geq 8|D\psi|_\infty|x| + 2|D^2\psi|_\infty|x|^2$ and $2C_2'/|x|^\gamma > 2|\mu||D\psi|/|x|$, for all $|x| \leq r_0$. Hence, from above we easily obtain

$$\begin{aligned} I^1_{l,x^2} \geq 2sC_2' &\iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big] \text{ dx dt} - C_4 s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \\ &- s \iint\limits_{\Omega\times(0,T)} \Delta\sigma_{x^2}\alpha|\nabla z|^2 \text{ dx dt} \end{aligned} \quad (4.3.17)$$

where $C_4 = 4C_3 + 2|\mu||D\psi|_\infty/r_0$.

Estimates for I^1_{l,σ_ϕ}:

In order to get rid of the gradient term with negative sign in (4.3.17) we have to estimate from below the quantity $\mathcal{T} := I^1_{l,\sigma_\phi} - s \iint\limits_{\Omega\times(0,T)} \Delta\sigma_{x^2}\alpha|\nabla z|^2 \text{ dx dt}$ that is

$$\mathcal{T} = -2s \iint\limits_{\Omega\times(0,T)} D^2\sigma_\phi(\nabla z, \nabla z) \text{ dx dt} - s \iint\limits_{\Omega\times(0,T)} \Delta\sigma\alpha|\nabla z|^2 \text{ dx dt} + 2\mu s \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^4} x\cdot\nabla\sigma_\phi \text{ dx dt}.$$

To do that, according to Lemma 4.3.1, formulas (4.3.6)-(4.3.7) and (4.2.5)-(4.2.9), we firstly observe the pointwise inequalities

$$2D^2\tau_\phi(\nabla z, \nabla z) + \Delta\tau\alpha|\nabla z|^2 \geq C_5\lambda\Big(\frac{|x|}{r_0}\Big)^{\lambda-2}\phi|\nabla z|^2, \quad \forall x \in \Omega_{r_0} \quad (4.3.18)$$

$$\big|2D^2\tau_\phi(\nabla z, \nabla z) + \Delta\tau\alpha|\nabla z|^2\big| \leq C_6\lambda^2\Big(\frac{|x|}{r_0}\Big)^{\lambda}\phi|\nabla z|^2, \quad \forall\ x \in \omega_0. \quad (4.3.19)$$

Since (4.2.2) is accomplished we get

$$2D^2\tau_\phi(\nabla z,\nabla z)+\Delta\tau\alpha|\nabla z|^2 \geq C_7\lambda\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2, \quad \forall x\in\mathcal{O}. \tag{4.3.20}$$

for some positive constants C_5, C_6, C_7 depending on ψ. On the other hand, it holds that

$$\frac{2\mu|x\cdot\nabla\tau_\phi|}{|x|^4} \leq C_8\lambda\big(\frac{|x|}{r_0}\big)^{\lambda-4}\phi, \quad \forall x\in\Omega. \tag{4.3.21}$$

where $C_8 = C_8(\psi,\mu) > 0$. Therefore, it follows from above that

$$\begin{aligned}\mathcal{T} \geq C_7 s\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}+ \\ + C_5 s\lambda \iint\limits_{\Omega_{r_0}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-2}|\nabla z|^2 \text{ dx dt} - C_8 s\lambda \iint\limits_{\Omega\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-4}\phi|z|^2 \text{ dx dt} \\ - C_6 s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}\end{aligned} \tag{4.3.22}$$

Summing the terms in (4.3.17)-(4.3.22) we get

$$\begin{aligned}I_l^1 \geq C_2 s \iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big] \text{ dx dt} + C_7 s\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}+ \\ + C_5 s\lambda \iint\limits_{\Omega_{r_0}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-2}|\nabla z|^2 \text{ dx dt} - C_8 s\lambda \iint\limits_{\Omega\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-4}\phi|z|^2 \text{ dx dt} \\ - C_4 s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} - C_6 s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda\phi|\nabla z|^2 \text{ dx dt}\end{aligned} \tag{4.3.23}$$

Estimates for I_l^2.

Making use of the support of α located far from origin we note that

$$|\Delta^2\tau_\phi|, \quad |\Delta\tau_\phi|, \quad |\nabla\Delta\tau_\phi|, \quad \Big|\alpha\frac{\Delta\tau_\phi}{|x|^2}\Big| \leq A_\lambda, \quad \forall x\in\Omega, \tag{4.3.24}$$

where A_λ is a big enough constant depending on λ. Then we get

$$I_l^2 \geq -A_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \tag{4.3.25}$$

Next we conclude

$$
\begin{aligned}
I_l \geq C_2 s \iint\limits_{\Omega\times(0,T)} \theta\Big[|x|^{2-\gamma}|\nabla z|^2 + \frac{z^2}{|x|^\gamma}\Big] \text{ dx dt} + C_7 s\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda \phi|\nabla z|^2 \text{ dx dt}+ \\
+ C_5 s\lambda \iint\limits_{\Omega_{r_0}\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^{\lambda-2}|\nabla z|^2 \text{ dx dt} - B_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \\
- C_6 s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\big(\frac{|x|}{r_0}\big)^\lambda \phi|\nabla z|^2 \text{ dx dt}, \quad (4.3.26)
\end{aligned}
$$

where $B_\lambda = C_4 + A_\lambda + C_8\lambda \sup_{x\in\Omega}\{(|x|/r_0)^\lambda\phi\}$.

□

Proof of Lemma 4.2.4. We split $I_{nl} = I_{nl,1} + I_{nl,2}$, where $I_{nl,1}$ are the integrals in I_{nl} restricted to Ω_{r_0} and $I_{nl,2}$ are the terms in I_{nl} restricted to $\tilde{\mathcal{O}}$. We put $\sigma = -\theta\tau$. Then I_{nl} could be written as

$$
\begin{aligned}
I_{nl} = 2s^3 \iint\limits_{Q_T} \theta^3|z|^2 D^2\tau(\nabla\tau,\nabla\tau) \text{ dx dt} - s^3 \iint\limits_{Q_T} \theta^3|z|^2\alpha\Delta\tau|\nabla\tau|^2 \text{ dx dt} \\
- \frac{s^2}{2} \iint\limits_{Q_T} \theta^2\alpha^2|z|^2|\Delta\tau|^2 \text{ dx dt}. \quad (4.3.27)
\end{aligned}
$$

The term I_{nl} is even more difficult to deal with because of technical computations. First of all we have

$$
\begin{aligned}
\partial_{x_i}\tau &= 2x_i\psi + |x|^2\partial_{x_i}\psi + \big(\frac{1}{r_0}\big)^\lambda\big(\lambda x_i|x|^{\lambda-2} + \lambda|x|^\lambda\partial_{x_i}\psi\big)\phi, \\
\partial^2_{x_ix_j}\tau &= 2\delta_{ij}\psi + 2x_i\partial_{x_j}\psi + 2x_j\partial_{x_i}\psi + |x|^2\partial^2_{x_ix_j}\psi + \big(\frac{1}{r_0}\big)^\lambda\Big[\lambda\delta_{ij} + \lambda(\lambda-2)x_ix_j|x|^{\lambda-4} + \\
&+ \lambda^2 x_j\partial_{x_i}\psi|x|^{\lambda-2} + \lambda^2 x_i\partial_{x_j}\psi|x|^{\lambda-2} + \lambda|x|^\lambda\partial_{x_ix_j}\psi + \lambda^2|x|^\lambda\partial_{x_i}\psi\partial_{x_j}\psi\Big]\phi. \quad (4.3.28)
\end{aligned}
$$

Using the expressions in (4.3.28) we obtain several useful formulas:

$$
\begin{aligned}
|x\cdot\nabla\tau|^2 &= |x|^2|\nabla\tau|^2 + \Big(|\nabla\psi\cdot x|^2 - |x|^2|\nabla\psi|^2\Big)\Big(|x|^2 + \lambda\big(\frac{1}{r_0}\big)^\lambda\phi|x|^\lambda\Big)^2, \\
(x\cdot\nabla\tau)(\nabla\psi\cdot\nabla\tau) &= |\nabla\tau|^2\nabla\psi\cdot x + \Big(|x|^2|\nabla\psi|^2 - |x\cdot\nabla\psi|^2\Big)\Big(1 + \lambda\big(\frac{1}{r_0}\big)^\lambda\phi|x|^{\lambda-2}\Big)\times \\
&\times\Big(2\psi|x|^2 + \lambda\big(\frac{1}{r_0}\big)^\lambda\phi|x|^\lambda\Big), \\
|\nabla\psi\cdot\nabla\tau|^2 &= |\nabla\psi|^2|\nabla\tau|^2 + \Big(|x\cdot\nabla\psi|^2 - |x|^2|\nabla\psi|^2\Big)\Big(|x|^2 + \lambda\big(\frac{1}{r_0}\big)^\lambda\phi|x|^\lambda\Big)^2. \quad (4.3.29)
\end{aligned}
$$

Using the expressions in (4.3.29) we reach to the identity

$$\begin{aligned}
&2D^2\tau(\nabla\tau,\nabla\tau)-\alpha\Delta\tau|\nabla\tau|^2=\\
&\quad=2\psi(2-\alpha N)|\nabla\tau|^2+4(2-\alpha)|\nabla\tau|^2\nabla\psi\cdot x+2|x|^2D^2\psi(\nabla\tau,\nabla\tau)+\\
&\quad+8(|x|^2|\nabla\psi|^2-|\nabla\psi\cdot x|^2)\Big(1+\lambda(\frac{1}{r_0})^\lambda\phi|x|^{\lambda-2}\Big)\Big(2\psi|x|^2+\lambda(\frac{1}{r_0})^\lambda\phi|x|^\lambda\Big)-\\
&\quad-\alpha|x|^2\Delta\psi|\nabla\tau|^2+(\frac{1}{r_0})^\lambda\phi\Big\{\Big[\Big((2-\alpha)\lambda^2-\lambda(2+\alpha N-2\alpha)\Big)|x|^{\lambda-2}+\\
&\quad+2\lambda^2(2-\alpha)|x|^{\lambda-2}\nabla\psi\cdot x+(2-\alpha)\lambda^2|x|^{\lambda+2}|\nabla\psi|^2\Big]|\nabla\tau|^2+\\
&\quad+(2-\alpha)\lambda|x|^\lambda D^2\psi(\nabla\tau,\nabla\tau)+(|x|^2|\nabla\psi|^2-|\nabla\psi\cdot x|^2)\times\\
&\quad\times\Big[4\lambda^3(\frac{1}{r_0})^{2\lambda}\phi^2|x|^{3\lambda-4}+8\lambda^2(\frac{1}{r_0})^\lambda\phi|x|^{2\lambda-2}+\\
&\quad+8\lambda^2\psi(1-\psi)|x|^\lambda-2\lambda(\lambda-2)|x|^\lambda\Big]\Big\}.
\end{aligned}$$

Computations for $I_{nl,1}$.

Taking into account the properties of ψ and α and Lemma 4.3.1 we can obtain

$$2D^2\tau(\nabla\tau,\nabla\tau)-\alpha\Delta\tau|\nabla\tau|^2\geq|\nabla\tau|^2+\lambda^2(\frac{1}{r_0})^\lambda\phi|x|^{\lambda-2}|\nabla\tau|^2,\quad\forall x\in\Omega_{r_0}. \tag{4.3.30}$$

On the other hand we have

$$C_9|x|^2\geq|\nabla\tau|^2\geq|x|^2,\quad\forall x\in\Omega_{r_0},$$

for some constant $C_9=C_9(\psi)>0$, and therefore

$$\begin{aligned}
2D^2\tau(\nabla\tau,\nabla\tau)-\alpha\Delta\tau|\nabla\tau|^2\alpha^2&\geq|x|^2,\quad\forall x\in\Omega_{r_0},\\
\alpha^2|\nabla\tau|^2&\leq C_9|x|^2,\quad\forall x\in\Omega_{r_0}.
\end{aligned}\tag{4.3.31}$$

In particular we obtain

$$I_{nl,1}\geq\frac{s^3}{2}\iint\limits_{\Omega_{r_0}\times(0,T)}\theta^3|x|^2|z|^2\ \mathrm{dx\ dt}. \tag{4.3.32}$$

for s large enough.

Computations for $I_{nl,2}$.

Again, according to the properties of ψ and α we observe that

$$\begin{aligned}
2D^2\tau(\nabla\tau,\nabla\tau)-\alpha\Delta\tau|\nabla\tau|^2&\geq C_{10}\lambda^2(\frac{1}{r_0})^\lambda\phi|x|^\lambda|\nabla\tau|^2\\
&\geq C_{11}\lambda^4(\frac{1}{r_0})^{3\lambda}|x|^{3\lambda}\phi^3,\quad\forall x\in\mathcal{O},\\
\alpha^2|\Delta\tau|^2&\leq C_{12}\lambda^4(\frac{1}{r_0})^{2\lambda}\phi^2|x|^{2\lambda},\quad\forall x\in\Omega
\end{aligned}\tag{4.3.33}$$

$$\begin{aligned}
\big|2D^2\tau(\nabla\tau,\nabla\tau)-\alpha\Delta\tau|\nabla\tau|^2\big|&\leq C_{13}\lambda^4(\frac{1}{r_0})^\lambda\phi|x|^\lambda|\nabla\tau|^2\\
&\leq C_{14}\lambda^4(\frac{1}{r_0})^{3\lambda}|x|^{3\lambda}\phi^3,\quad\forall x\in\omega_0
\end{aligned}\tag{4.3.34}$$

for some uniform constants $C_{10}, C_{11}, C_{12}, C_{13}, C_{14}$ depending only on ψ. Then we obtain

$$I_{nl,2} \geq C_{10} s^3 \lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} \theta^3 \big(\frac{1}{r_0}\big)^{3\lambda} \phi^3 |x|^{3\lambda} |z|^2 \text{ dx dt} - C_{14} s^3 \lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3 \big(\frac{1}{r_0}\big)^{3\lambda} \phi^3 |x|^{3\lambda} |z|^2 \text{ dx dt} - \\ - C_{12} s^2 \lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^2 \big(\frac{1}{r_0}\big)^{2\lambda} |x|^{2\lambda} \phi^2 |z|^2 \text{ dx dt}. \quad (4.3.35)$$

By summing the terms in (4.3.35)-(4.3.32) we obtain

$$I_{nl} \geq \frac{s^3}{2} \iint\limits_{\Omega_{r_0}\times(0,T)} \theta^3 |x|^2 |z|^2 \text{ dx dt} + C_{15} s^3 \lambda^4 \iint\limits_{\mathcal{O}\times(0,T)} \theta^3 \big(\frac{1}{r_0}\big)^{3\lambda} \phi^3 |x|^{3\lambda} |z|^2 \text{ dx dt} \\ - C_{16} s^3 \lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3 \big(\frac{1}{r_0}\big)^{3\lambda} \phi^3 |x|^{3\lambda} |z|^2 \text{ dx dt} \quad (4.3.36)$$

where $C_{15} = C_{10}/2$ and $C_{16} = C_{12} + C_{14}$.

□

Proof of Lemma 4.2.5. According to the expression of θ we obtain

$$|\theta'| \leq C\theta^{1+1/k}, \quad |\theta''| \leq C\theta^{1+2/k},$$

for some positive constant C. On the other hand according to the definition of σ we get

$$\begin{aligned} |\Delta\sigma| &\leq D_\lambda \theta, \quad \forall \quad x \in \Omega, \\ |\partial_t \sigma| &\leq D_\lambda \theta', \quad \forall \quad x \in \Omega, \\ \partial_t(|\nabla\sigma|^2) &\leq D_\lambda \theta\theta' |x|^2, \quad \forall \quad x \in \Omega_{r_0}, \\ \partial_t(|\nabla\sigma|^2) &\leq D_\lambda \theta\theta' |x|^{2\lambda}, \quad \forall \quad x \in \tilde{\mathcal{O}}, \end{aligned} \quad (4.3.37)$$

for a big enough constant $D_\lambda > 0$ depending on λ. Since α is supported far from origin we can write

$$s^2 \iint\limits_{\Omega\times(0,T)} \Big|\alpha |z|^2 \Delta\sigma \partial_t \sigma\Big| \text{ dx dt} \leq \frac{4D_\lambda^2}{r_0^2} s^2 \iint\limits_{\Omega_{r_0}\times(0,T)} \theta^{2+1/k} |x|^2 |z|^2 \text{ dx dt} \\ + D_\lambda^2 s^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^{2+1/k} \, |z|^2 \text{ dx dt} \quad (4.3.38)$$

From (5.3.34) we also obtain

$$s^2 \iint\limits_{\Omega\times(0,T)} \Big| |z|^2 \partial_t(|\nabla\sigma|^2)\Big| \text{ dx dt} \leq D_\lambda s^2 \iint\limits_{\Omega_{r_0}\times(0,T)} \theta^{2+1/k} |x|^2 |z|^2 \text{ dx dt} \\ + D_\lambda s^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^{2+1/k} |x|^{2\lambda} |z|^2 \text{ dx dt}. \quad (4.3.39)$$

Now, we put

$$R := -\frac{s}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\partial_{tt}\sigma \text{ dx dt} - B_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt},$$

where B_λ is elected as in Lemma 4.2.3.

Then, there exists $E_\lambda > 0$ such that

$$\begin{aligned}\Big|\frac{s}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\partial_{tt}^2\sigma \text{ dx dt}\Big| &\le E_\lambda s \iint\limits_{\Omega\times(0,T)} \theta^{1+2/k}|z|^2 \text{ dx dt}\\ \Big|B_\lambda s\iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt}\Big| &\le E_\lambda s \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt}. \end{aligned}\tag{4.3.40}$$

Summing these bounds we obtain

$$|R| \le 2sE_\lambda \iint\limits_{\Omega\times(0,T)} \theta^{1+2/k}|z|^2 dx. \tag{4.3.41}$$

In the sequel, we write

$$\iint\limits_{Q_T} \theta^{1+2/k}|z|^2 \text{ dx dt} = \iint\limits_{Q_T} \Big(\beta\theta^{1+2/k-1/q'}|x|^{\gamma/q'}|z|^{2/q'}\Big)\Big(\frac{1}{\beta}\theta^{1/q'}|x|^{-\gamma/q'}|z|^{2/q'}\Big) \text{ dx dt}.$$

Now we take

$$q = \frac{2+\gamma}{\gamma}, \quad q' = \frac{\gamma+2}{2}.$$

Note that $1/q + 1/q' = 1$ and applying the Young inequality we obtain

$$\begin{aligned}\iint\limits_{Q_T} \theta^{1+2/k}|z|^2 \text{ dx dt} &\le \beta^q \iint\limits_{Q_T} \theta^{(1+2/k-1/q')q}|x|^2|z|^2 dxdt + \frac{1}{\beta^{q'}}\iint\limits_{Q_T} \theta\frac{|z|^2}{|x|^\gamma} \text{ dx dt}\\ &= \beta^q \iint\limits_{Q_T} \theta^3|x|^2|z|^2 \text{ dx dt} + \frac{1}{\beta^{q'}}\iint\limits_{Q_T} \theta\frac{|z|^2}{|x|^\gamma} \text{ dx dt}, \end{aligned}\tag{4.3.42}$$

provided $k = 1 + 2/\gamma$. Therefore, from (4.3.41) and (4.3.42) we have

$$|R| \le C(\lambda)s\Big(\beta^q \iint\limits_{Q_T} \theta^3|x|^2|z|^2 \text{ dx dt} + \frac{1}{\beta^{q'}}\iint\limits_{Q_T} \theta\frac{|z|^2}{|x|^\gamma} \text{ dx dt}\Big). \tag{4.3.43}$$

Consequently, from (4.3.38)-(4.3.43) follows

$$\begin{aligned}|I_r| \le F_\lambda\Big(s^2 &\iint\limits_{\Omega_{r_0}\times(0,T)} \theta^2|x|^2|z|^2 \text{ dx dt} + s\beta^q \iint\limits_{Q_T} \theta^3|x|^2|z|^2 \text{ dx dt}\\ &+ \frac{s}{\beta^{q'}}\iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|^\gamma} \text{ dx dt} + s^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3|x|^{3\lambda}|z|^2 \text{ dx dt}\Big), \end{aligned}\tag{4.3.44}$$

for a new constant $F_\lambda > 0$.

Take β such that $F_\lambda/\beta^{q'} = C_2/2$. Then there exists some $s_0(\lambda)$ such that for $s \ge s_0(\lambda)$ we finish the proof of Lemma 4.2.5. $\square$

4.4. Sketch of the proof of Theorem 4.1.2

Let us also assume $\omega \subset \Omega$ is a non-empty control region such that $\Omega \cap \overline{B_1(0)}$ is included in Ω and $\overline{B_1(0)} \cap \bar{\omega}$ is empty. This can always be done by a scaling argument. Moreover, let us also consider a non-empty open subset $\omega_0 \subset\subset \omega$.

Again, the proof is based on Carleman estimates as shown in Theorem 4.1.1. Before sketching the proof it is worthy to make some remarks about the technical difficulties that appear for domains Ω not satisfying the property (P) (as in the hypothesis of Theorem 4.1.2) when choosing the weight ψ.

If it were possible to construct a weight ψ as in Theorem 4.1.1, for a domain which does not satisfy the property (P), then the result of Theorem 4.1.1 could be generalized to any geometrical configuration of Ω in the sense that we could control our system with a distributed control in any open subset $\omega \subset \Omega$ no matter what the geometry of Ω is. Unfortunately, for a domain Ω as in the hypothesis of Theorem 4.1.2 we have not found any example of such ψ as in Subsection 4.2.1, since the distance function $d = d(x, \partial\Omega)$ mentioned in Remark 4.2.1 violates the condition (4.2.9) in this geometrical configuration.

Despite of this, for the proof of Theorem 4.1.2 (Ω not satisfying the property (P)) we choose the weight ψ to satisfy the same constraints as in Subsection 4.2.1 less the condition (4.2.9). In view of that, the result of Theorem 4.1.2 is weaker than the result provided by Theorem 4.1.1. Here we skip the details of the proof of Theorem 4.1.2 since it is straightforward from the Carleman estimates developed for Theorem 4.1.1, pointing out just the conclusions.

Indeed, when doing Carleman estimates, because the loss of condition (4.2.9), we cannot capture the energy in the annulus region

$$\omega' := \{x \in \Omega \mid r_0 < |x| < r_1\}, \tag{4.4.1}$$

where r_0 stands for the constant in (4.2.5) (r_0 is small enough since it is proportional with $1/|\nabla\psi|$) and r_1 is any fixed positive constant such that $r_1 \leq 1$. For that reason, we succeed to control our system forcing the control to act precisely in $\omega \cup \omega'$, where ω' is defined in (4.4.1). If $\omega_0^{'}$ is a non-empty open subset $\omega_0^{'} \subset\subset \omega'$ we obtain the following Carleman estimate:

Theorem 4.4.1. *There exists a positive constant K_1 and λ_0 such that for $\lambda \geq \lambda_0$ there exists $s_0(\lambda)$ such that for any $s \geq s_0$ we have*

$$\begin{aligned}
&s\lambda^2 \iint_{\mathcal{O}\times(0,T)} \theta\Big(\frac{|x|}{r_0}\Big)^{\lambda}\phi e^{-2s\sigma}|\nabla w|^2 \,\mathrm{dx}\,\mathrm{dt} + s \iint_{\Omega\times(0,T)} \theta e^{-2s\sigma}\Big[|x|^{2-\gamma}|\nabla w|^2 + \frac{w^2}{|x|^{\gamma}}\Big] \,\mathrm{dx}\,\mathrm{dt} \\
&\qquad + s^3 \iint_{\Omega\times(0,T)} \theta^3 e^{-2s\sigma}|x|^2|w|^2 \,\mathrm{dx}\,\mathrm{dt} + s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \theta^3 e^{-2s\sigma}\Big(\frac{|x|}{r_0}\Big)^{3\lambda}\phi^3|w|^2 \,\mathrm{dx}\,\mathrm{dt} \\
&\leq K\Big(s\lambda^2 \iint_{\omega_0\cup\omega_0^{'}\times(0,T)} \theta\Big(\frac{|x|}{r_0}\Big)^{\lambda}\phi e^{-2s\sigma}|\nabla w|^2 \,\mathrm{dx}\,\mathrm{dt} + s^3\lambda^4 \iint_{\omega_0\cup\omega_0^{'}\times(0,T)} \theta^3\Big(\frac{|x|}{r_0}\Big)^{3\lambda}\phi^3 e^{-2s\sigma}|w|^2 \,\mathrm{dx}\,\mathrm{dt}\Big).
\end{aligned} \tag{4.4.2}$$

In particular, according to the Cacciopoli's inequality, Theorem 4.4.1 yields the proof of Theorem 4.1.2.

Chapter 5

Regularity of the Schrödinger operator with one singular potential

Abstract. In this chapter we are dealing with the Poisson problem corresponding to the Schrödinger operator $A_\lambda := -\Delta - \lambda/|x|^2$ acting in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, containing the origin either on interior or on the boundary. We give explicit formulas for the radial solutions and we study their regularity. This will be justified in any dimension $N \geq 1$, $N \neq 2$, but the key point is played by the one dimensional case ($N = 1$), since the multi-dimensional case ($N \geq 3$) can be reduced to that one. In the non-radial case, we study the asymptotic behavior at $x = 0$ for the solutions since the singularity occurs at origin. This is done by means of spectral analysis and fine properties of the Bessel functions.

5.1. Introduction

Let us consider the N-dimensional singular elliptic problem

$$\begin{cases} -\Delta u - \lambda \frac{u}{|x|^2} = f & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5.1.1}$$

Throughout this chapter, for system (5.1.1) we assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open bounded subset such that the singularity $x = 0$ is located either in the interior (when $N \geq 3$) or on the boundary (when $N = 1$). Moreover, we assume that the applied force f belongs to $L^2(\Omega)$ and $\lambda \leq \lambda_\star = (N-2)^2/4$, where $\lambda_\star$ is the best constant in the Hardy inequality before (see (2.1.5), Chapter 2). Besides, we denote by H_λ, the Hilbert space induced by the Hardy quadratic form as defined in Chapter 3. In addition, we recall that $H_\lambda = H_0^1(\Omega)$, for any subcritical value $\lambda < \lambda_\star$ whereas in the critical case $\lambda = \lambda_\star$, $H_{\lambda_\star}$ is strictly larger than $H_0^1(\Omega)$.

The operator $A_\lambda := -\Delta - \lambda/r^2$ is coercive in H_λ provided $\lambda \leq \lambda_\star$, due to (2.1.5). By the classical Lax-Milgram Lemma, there exists a unique variational solution $u \in H_\lambda$ for problem (5.1.1) i.e. u solves

$$\begin{cases} (u, v)_{H_\lambda, H_\lambda} = (f, v)_{L^2(0,1), L^2(0,1)}, & \forall v \in H_\lambda, \\ u \in H_\lambda, \end{cases} \tag{5.1.2}$$

where $(\cdot,\cdot)_X$ denotes the scalar product induced by the space X. This makes the problem (5.2.1) to be well-posed in H_λ. Here we address the question of possible "gain" of regularity to (5.2.1) provided $f \in L^2(\Omega)$.

Aspects of the regularity properties play a crucial role in the study of PDEs. This issue been intensively analyzed for a large class of elliptic and evolution operators.

The regularity properties for a general second order elliptic operator turn out to be related to both the regularity of the domain Ω and regularity of the coefficients involved in the operator.

For the Laplacian $-\Delta$ (which corresponds to $\lambda = 0$) there are some classical results as follows. If Ω is a convex domain then the solution of (5.1.1) for $\lambda = 0$ satisfies $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Otherwise, if Ω is not convex, there is a lack of regularity at the points where Ω loses the convexity property. For more concrete examples in which there is a lack of regularity to the Laplace operator (domains with cracks, corners, etc.) we refer to the book by Grisvard [76] and the references therein.

Note that coefficients of the equation in (5.1.1) blow-up at the origin (unless $\lambda = 0$) and therefore standard elliptic regularity does not apply in this case. Recently, during the redaction of this Thesis, we have been awared of a paper by Peral et al. [17] who discussed some optimal regularity in $W^{1,q}$, $q > 1$ for the problem $A_\lambda u = f$, where q depends on the L^p-norm of the data f. In this chapter we discuss regularity properties for A_λ in the Hilbert space H^{1+s}, $s \geq 0$, in the non-trivial case $\lambda \neq 0$. We discuss the one dimensional case ($N = 1$) and multi-dimensional case ($N \geq 3$) separately.

5.2. The one dimensional case

Let us consider the one dimensional singular elliptic problem

$$\begin{cases} -u''(r) - \lambda \frac{u(r)}{r^2} = f(r), \quad r \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.2.1}$$

where the applied force f belongs to $L^2(0,1)$, $\lambda \leq 1/4$, and $1/4$ is the best constant in the 1-d Hardy inequality (2.1.1).

In the following we determine explicit formulas for the solution of (5.2.1). We split the analysis in two cases by treating the subcritical case $\lambda < 1/4$ and the critical case $\lambda = 1/4$ separately.

5.2.1. Main results

Given $f \in L^2(0,1)$, let us introduce the functions

$$G_1(r) = \int_1^r t^{\frac{1}{2}-\sqrt{1/4-\lambda}} f(t)dt, \tag{5.2.2}$$

$$G_2(r) = \int_1^r t^{\frac{1}{2}+\sqrt{1/4-\lambda}} f(t)dt, \tag{5.2.3}$$

which are well defined for any $r > 0$, respectively $r \geq 0$.

Next, we claim the main results of this section.

Theorem 5.2.1 (Subcritical case). *Assume $\lambda < 1/4$. The function*

$$u(r) = u_S + u_R,$$

where

$$\begin{aligned} u_S(r) &= \frac{1}{2\sqrt{1/4-\lambda}}(G_2(0) - G_1(r))r^{\frac{1}{2}+\sqrt{1/4-\lambda}}, \\ u_R(r) &= \frac{1}{2\sqrt{1/4-\lambda}}(G_2(r) - G_2(0))r^{\frac{1}{2}-\sqrt{1/4-\lambda}}, \end{aligned}$$

is the unique function $u \in H_0^1(0,1)$, solution of the problem (5.2.1).

Theorem 5.2.2 (Critical case). *Assume $\lambda = 1/4$. The unique solution $u \in H_{1/4}$ of problem*

$$\begin{cases} -u'' - \frac{u}{4r^2} = f, & r \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.2.4}$$

is given by

$$u(r) = \Big(\int_1^r t^{1/2}(\log t) f(t)dt\Big) r^{1/2} - \Big(\int_1^r t^{1/2} f(t)dt\Big) r^{1/2}\log r. \tag{5.2.5}$$

5.2.2. Preliminaries: $\lambda < 1/4$

In the sequel we prove Theorem 5.2.1. This is based on Lemma 5.2.1 whose proof, applying the method of variation of constants, is given at the end of Section 5.2.2.

Lemma 5.2.1. *Assume $\lambda < 1/4$ and $f \in L^2(0,1)$. The family of solutions of the singular ODE*

$$-u'' - \frac{\lambda}{r^2}u = f, \quad r \in (0,1) \tag{5.2.6}$$

is given by

$$\begin{aligned} u(r) = {} & C_1 r^{\frac{1}{2}+\sqrt{1/4-\lambda}} + C_2 r^{\frac{1}{2}-\sqrt{1/4-\lambda}} + \\ & + \Big[-\frac{1}{2\sqrt{1/4-\lambda}}\int_1^r t^{\frac{1}{2}-\sqrt{1/4-\lambda}} f(t)dt + K_1\Big] r^{\frac{1}{2}+\sqrt{1/4-\lambda}} \\ & + \Big[\frac{1}{2\sqrt{1/4-\lambda}}\int_1^r t^{\frac{1}{2}+\sqrt{1/4-\lambda}} f(t)dt + K_2\Big] r^{\frac{1}{2}-\sqrt{1/4-\lambda}}, \end{aligned} \tag{5.2.7}$$

where C_1, C_2, K_1, K_2 are real constants.

Proof of Theorem 5.2.1. By direct computations one can prove that u verifies the equation of (5.2.1) (being a particular solution in Lemma 5.2.1). We have to check that the zero boundary conditions and regularity $u \in H^1(0,1)$ are full-filled. Then such a solution $u \in H_0^1(0,1)$ is unique determined from Lax-Milgram's Theorem. Firstly, it is easy to see that $u(1) = 0$. However, it is not trivial that the trace of u vanishes at zero. Nevertheless, we can check that

$$\lim_{r\to 0} u(r) = 0,$$

by showing the convergence of the quantities

$$G_1(r)r^{\frac{1}{2}+\sqrt{1/4-\lambda}} \to 0, \quad (G_2(r)-G_2(0))r^{\frac{1}{2}-\sqrt{1/4-\lambda}} \to 0, \qquad \text{as } r \to 0. \tag{5.2.8}$$

Let us check the validity of (5.2.8). Indeed, if $\sqrt{1/4-\lambda} \neq 1$, using Hölder inequality we obtain

$$\begin{aligned} |G_1(r)| &\leq \Big(\int_r^1 t^{1-2\sqrt{1/4-\lambda}}dt\Big)^{\frac{1}{2}}||f||_{L^2(0,1)} \lesssim (1+r^{2-2\sqrt{1/4-\lambda}})^{\frac{1}{2}}||f||_{L^2(0,1)} \\ &\lesssim 1+r^{1-\sqrt{1/4-\lambda}}, \quad \forall r \quad \in (0,1). \end{aligned} \tag{5.2.9}$$

Otherwise, if $\sqrt{1/4-\lambda}=1$, then

$$|G_1(r)| \leq \big(\int_r^1 t^{-1}dt\big)^{\frac{1}{2}}||f||_{L^2(0,1)} \lesssim \sqrt{\log\frac{1}{r}}, \tag{5.2.10}$$

Thus

$$|G_1(r)|r^{\frac{1}{2}+\sqrt{1/4-\lambda}} \lesssim \begin{cases} r^{\frac{1}{2}+\sqrt{1/4-\lambda}}+r^{\frac{3}{2}}, & \sqrt{1/4-\lambda}\neq 1, \\ r^{\frac{3}{2}}\sqrt{\log\frac{1}{r}}, & \sqrt{1/4-\lambda}=1, \end{cases} \tag{5.2.11}$$

and therefore $\lim_{r\to 0} G_1(r)r^{1+2\sqrt{1/4-\lambda}}=0$ in both of the cases in (5.2.11).

We are checking now the second limit in (5.2.8). Firstly,

$$|G_2(r)-G_2(0)| \leq \int_0^r t^{1+2\sqrt{1/4-\lambda}}dt||f||_{L^2(0,1)} \lesssim r^{1+\sqrt{1/4-\lambda}}, \tag{5.2.12}$$

and in consequence

$$\lim_{r\to 0}|G_2(r)-G_2(0)|r^{\frac{1}{2}-\sqrt{1/4-\lambda}} \lesssim \lim_{r\to 0} r^{\frac{3}{2}}=0.$$

So the boundary conditions are full-filled. It remains to check that $u \in H^1(0,1)$. In that sense, we note that

$$u'(r)=I_1(r)+I_2(r)+I_3(r), \tag{5.2.13}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\sqrt{1/4-\lambda}}\Big(\frac{1}{2}+\sqrt{1/4-\lambda}\Big)G_2(0)r^{-\frac{1}{2}+\sqrt{1/4-\lambda}}, \\ I_2 &= -\frac{1}{2\sqrt{1/4-\lambda}}\Big(\frac{1}{2}+\sqrt{1/4-\lambda}\Big)G_1(r)r^{-\frac{1}{2}+\sqrt{1/4-\lambda}}, \\ I_3 &= \frac{1}{2\sqrt{1/4-\lambda}}\Big(\frac{1}{2}-\sqrt{1/4-\lambda}\Big)(G_2(r)-G_2(0))r^{-\frac{1}{2}-\sqrt{1/4-\lambda}}. \end{aligned}$$

Obviously $I_1(r) \in L^2(0,1)$. For I_2 we have from (5.2.9), (5.2.10) that

$$|I_2| \lesssim \begin{cases} r^{-\frac{1}{2}+\sqrt{1/4-\lambda}}+r^{\frac{1}{2}}, & \sqrt{1/4-\lambda}\neq 1, \\ r^{\frac{1}{2}}\sqrt{\log\frac{1}{r}}, & \sqrt{1/4-\lambda}=1, \end{cases} \tag{5.2.14}$$

which implies $I_2 \in L^2(0,1)$. For the third term I_3 one can obtain from (5.2.12) that

$$|I_3(r)| \lesssim r^{\frac{1}{2}},$$

that also belongs to $L^2(0,1)$. We conclude that $u \in H^1(0,1)$ and Lemma 5.2.1 yields true.

$\square$

Proof of Lemma 5.2.1. In order to subtract the singularity at $r = 0$ we propose the change of variable

$$u(r) = v(\log r) = v(t), \quad r = e^t.$$

Then, the equation (5.2.6) is converted into

$$-v'' + v' - \lambda v = g(t), \quad x \in (-\infty, 0), \tag{5.2.15}$$

where

$$g(t) = e^{2t} f(e^t).$$

Next it suffices to solve (5.2.15) which is an non-homogeneous second order differential equation with constant coefficients. We proceed as follows by means of the method of variation of constants. Firstly we solve the homogenous equation of (5.2.15)

$$-v_g^{''} + v_g^{'} - \lambda v_g = 0, \tag{5.2.16}$$

and we find the family of solutions

$$v_g = C_1 e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} + C_2 e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t}, \quad C_1, C_2 \in \mathbb{R}.$$

Next we are looking for a particular solution to the equation (5.2.15), of the form

$$v_p(t) = C_1(t) e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} + C_2(t) e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t}. \tag{5.2.17}$$

Then, the functions $C_1(t)$, $C_2(t)$ must satisfy the system

$$\begin{cases} C_1'(t) e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} + C_2'(t) e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t} = 0, \\ C_1'(t)\left(\frac{1}{2} + \sqrt{1/4-\lambda}\right) e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} + C_2'(t)\left(\frac{1}{2} - \sqrt{1/4-\lambda}\right) e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t} = g(t). \end{cases}$$

The Wronskian, which is the determinant of the above system, is equal to

$$W = \left\| \begin{array}{cc} e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} & e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t} \\ \left(\frac{1}{2} + \sqrt{1/4-\lambda}\right) e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} & \left(\frac{1}{2} - \sqrt{1/4-\lambda}\right) e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t} \end{array} \right\| = 2\sqrt{1/4-\lambda}\, e^t. \tag{5.2.18}$$

After solving the system and integrating we obtain

$$C_1(t) = -\frac{1}{2\sqrt{1/4-\lambda}} \int_0^t e^{\left(-\frac{1}{2}-\sqrt{1/4-\lambda}\right)s} g(s)ds + K_1, \quad K_1 \in \mathbb{R},$$

$$C_2(t) = \frac{1}{2\sqrt{1/4-\lambda}} \int_0^t e^{\left(-\frac{1}{2}+\sqrt{1/4-\lambda}\right)s} g(s)ds + K_2, \quad K_2 \in \mathbb{R},$$

The general solution of (5.2.15) is determined by the sum of the general solution v_g of the homogeneous equation (5.2.16) and the particular solution v_p with variable coefficients in (5.2.17) to the nonhomogeneous equation (5.2.15). Due to this and (5.2.18) we obtain

$$\begin{aligned}v(t) = v_g(t) + v_p(t) = C_1 e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} + C_2 e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t} + \\ + \Big[-\frac{1}{2\sqrt{1/4-\lambda}}\int_0^t e^{\left(-\frac{1}{2}-\sqrt{1/4-\lambda}\right)s} g(s)ds + K_1\Big] e^{\left(\frac{1}{2}+\sqrt{1/4-\lambda}\right)t} \\ + \Big[\frac{1}{2\sqrt{1/4-\lambda}}\int_0^t e^{\left(-\frac{1}{2}+\sqrt{1/4-\lambda}\right)s} g(s)ds + K_2\Big] e^{\left(\frac{1}{2}-\sqrt{1/4-\lambda}\right)t},\end{aligned}$$

where C_1, C_2, K_1, K_2 are real numbers. Undoing the variables we obtain the general solution of the equation

$$-u'' - \frac{\lambda}{r^2}u = f$$

as stated in (5.2.7). □

5.2.3. The critical value $\lambda = 1/4$

The goal of this section consists to prove Theorem 5.2.2. This is based on the following preliminary lemma.

Lemma 5.2.2. *Given $f \in L^2(0,1)$, the family of distributional solutions of the singular ODE*

$$-u'' - \frac{1}{4r^2}u = f, \quad r \in (0,1) \tag{5.2.19}$$

is given by

$$\begin{aligned}u(r) = C_1 r^{1/2} + C_2 r^{1/2}\log r + \Big[\int_1^r t^{1/2}(\log t) f(t)dt + K_1\Big] r^{1/2} \\ + \Big[-\int_1^r t^{1/2} f(t)dt + K_2\Big] r^{1/2}\log r,\end{aligned} \tag{5.2.20}$$

where C_1, C_2, K_1, K_2 are real constants.

The proof of Lemma 5.2.2 is postponed at the end of Section 5.2.3.

Proof of Theorem 5.2.2. The unique solution $u \in H_{1/4}$ to problem (5.2.4) is a particular solution in Lemma 5.2.2. Since u satisfies the boundary conditions $u(0) = u(1) = 0$ we necessary have $C_1 + K_1 = 0$ in the general formula (5.3.11). Therefore, the problem reduces to finding the constant C such that

$$\begin{aligned}u(r) = C r^{1/2}\log r + \Big(\int_1^r t^{1/2}(\log t) f(t)dt\Big) r^{1/2} \\ + \Big(-\int_1^r t^{1/2} f(t)dt\Big) r^{1/2}\log r,\end{aligned} \tag{5.2.21}$$

belongs to $u \in H_{1/4}$.

Setting $\phi = r^{1/2}\log r$, note that $\phi \notin H_{1/4}$ since

$$||\phi||^2_{H_{1/4}} = \int_0^1 (\phi' - \frac{\phi}{2r})^2 dr = \int_0^1 r^{-1} dr = \infty. \tag{5.2.22}$$

Therefore, necessary we must have $C = 0$ in (5.2.21). Next, one can easily check that

$$u = \Big(\int_1^r t^{1/2}(\log t) f(t)dt\Big) r^{1/2} + \Big(-\int_1^r t^{1/2} f(t)dt\Big) r^{1/2}\log r,$$

satisfies the variational formulation of problem (5.2.4) in $H_{1/4}$ and the proof finishes.

□

Proof of Lemma 5.2.2. In order to subtract the singularity at $r = 0$ we propose the change of variable

$$u(r) = v(\log r) = v(t), \quad r = e^t.$$

Then, the problem (5.2.19) is converted into

$$-v'' + v' - \frac{1}{4}v = g(t), x \in (-\infty, 0), \tag{5.2.23}$$

where $g(t) = e^{2t}f(e^t)$. Next it suffices to solve (5.2.23) which is an inhomogeneous second order differential equation with constant coefficients. As in the subcritical case we proceed by means of the method of variation of constants. Firstly we solve the homogenous equation

$$-v_g^{''} + v_g^{'} - \frac{1}{4}v_g = 0, \tag{5.2.24}$$

and we find the family of solutions

$$v_g = C_1 e^{t/2} + C_2 t e^{t/2}, \quad C_1, C_2 \in \mathbb{R}.$$

Next we are looking for a particular solution to (5.2.23), of the form

$$v = C_1(t)e^{t/2} + C_2(t)te^{t/2}, \quad C_1, C_2 \in \mathbb{R}.$$

Then, the functions $C_1(t)$, $C_2(t)$ must satisfy the system

$$\begin{cases} C_1^{'}(t)e^{t/2} + C_2^{'}(t)te^{t/2} = 0, \\ C_1^{'}(t)\frac{1}{2}e^{t/2} + C_2^{'}(t)(\frac{t}{2}+1)e^{t/2} = -g(t). \end{cases}$$

The Wronskian, which is the determinant of the above system, is equal to

$$W = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & (\frac{t}{2}+1)e^{t/2} \end{vmatrix} = e^t.$$

After solving the system and integrating we obtain

$$C_1(t) = \int_0^t se^{3s/2} f(e^s)ds + K_1, \quad K_1 \in \mathbb{R},$$

$$C_2(t) = -\int_0^t e^{3s/2} f(e^s) ds + K_2, \quad K_2 \in \mathbb{R},$$

Then the general solution of (5.2.23) is given by the sum $v_g + v_p$. Therefore we have

$$v(t) = C_1 e^{t/2} + C_2 t e^{t/2} \quad + \Big[\int_0^t s e^{3s/2} f(e^s) ds + K_1 \Big] e^{t/2} + \Big[-\int_0^t e^{3s/2} f(e^s) ds + K_2 \Big] t e^{t/2},$$

where C_1, C_2, K_1, K_2 are real numbers. Undoing the variables we obtain the general solution of the equation

$$-u'' - \frac{1}{4r^2} u = f$$

as stated in (5.2.20). □

5.2.4. Regularity of A_λ

In order to study the regularity of A_λ we need to deal with Fractional Sobolev spaces. Next we recall some definitions and useful properties of these spaces (see e.g. [1]).

Definition 5.2.1. *Given $\Omega \subset \mathbb{R}^N$, $1 \leq p < \infty$ and $0 < s < 1$ one define the space $W^{s,p}(\Omega)$ constituting in the functions $u \in L^p(\Omega)$ for which the norm*

$$||u||^p_{W^{s,p}(\Omega)} := ||u||^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{sp+N}} dy dx. \tag{5.2.25}$$

is finite. It can be noted that $W^{s,p}(\Omega)$ is a Banach space with the norm $||\cdot||_{W^{s,p}(\Omega)}$ and defines $W^{s,p}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in this norm.

Definition 5.2.2. *Given $\Omega \subset \mathbb{R}^N$, $1 \leq p < \infty$, $0 < s < 1$, $m \in \mathbb{N}$ one define the Sobolev space $W^{m+s,p}(\Omega)$ constituting the functions $u \in W^{m,p}(\Omega)$ for which the norm*

$$||u||_{W^{m+s,p}(\Omega)} := \sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)} + \sum_{|\alpha| = m} ||D^\alpha u||_{W^{s,p}(\Omega)}. \tag{5.2.26}$$

Then $W^{m+s,p}(\Omega)$ is a Banach space with the norm $||\cdot||_{W^{m+s,p}(\Omega)}$ and defines the space $W_0^{m+s,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in this norm.

In particular, for radial functions of the form $u(x) = |x|^\alpha$, $\alpha \in \mathbb{R}$, where Ω is a bounded domain containing the origin, it holds that (see e.g. Pratelli [97])

$$u \in W^{m+s,p}(\Omega) \text{ iff } u \in W^{m,p}(\Omega) \text{ and } p(\alpha - m - s) > -N. \tag{5.2.27}$$

Moreover

$$|||x|^\alpha||^p_{W^{m+s,p}(\Omega)} = |||x|^\alpha||^p_{W^{m,p}(\Omega)} + \int_\Omega |\partial^{m+s}\big(|x|^\alpha\big)|^p dx, \tag{5.2.28}$$

where we have denoted symbolically the fractional derivative of s^{th} order by

$$\partial^{m+s}\big(|x|^\alpha\big) := \alpha x |x|^{\alpha - m - s + 1}.$$

Let be $1 \leq p \leq \infty$ and $0 \leq s \leq \infty$. By scaling arguments, the Sobolev exponent of $W^{s,p}(\Omega)$ is defined by the number

$$\mathrm{sob}(W^{s,p}) = s - \frac{N}{p}. \tag{5.2.29}$$

Next let us consider $0 \leq s_1 \leq s_2 \leq \infty$ and $1 \leq p_1, p_2 \leq \infty$. Then, cf. [97], the embedding

$$W^{s_1,p_1}(\Omega) \subset W^{s_2,p_2}(\Omega), \tag{5.2.30}$$

holds true if $\mathrm{sob}(W^{s_1,p_1}) \geq \mathrm{sob}(W^{s_2,p_2})$.

These basic results on Sobolev embeddings will be useful in the proof of the following regularity theorems.

Main results

The main result of this subsection states

Theorem 5.2.3 (Regularity). *Assume $f \in L^2(0,1)$, $\lambda \leq 1/4$ and u stands for the variational solution of* (5.2.1) *in H_λ. Then we have*

1. *If $\sqrt{1/4-\lambda} \leq 1$ then $u \in H^{1+s}(0,1)$, for all $s < \sqrt{1/4-\lambda}$.*
2. *If $\sqrt{1/4-\lambda} > 1$ then $u \in H^{1+s}(0,1)$, for all $s < 1$.*

Next we give one example to emphasize that Theorem 5.2.3 provides optimal regularity for those λ's satisfying $0 \leq \sqrt{1/4-\lambda} \leq 1$.

Example: Let $\varepsilon > 0$ aimed to be small and put $f(r) = r^{-\frac{1}{2}+\varepsilon}$, $r \in (0,1)$. Notice that $f \in L^2(0,1)$. Then

$$u(r) = -\frac{1}{\frac{3}{4}+\varepsilon^2+2\varepsilon+\lambda} r^{\frac{3}{2}+\varepsilon} + \frac{1}{\frac{3}{4}+\varepsilon^2+2\varepsilon+\lambda} r^{\frac{1}{2}+\sqrt{1/4-\lambda}}, \tag{5.2.31}$$

provided $\lambda \neq -(3/4+\varepsilon^2+2\varepsilon)$, $\lambda \leq 1/4$, is the solution to (5.2.1) in H_λ. Indeed, for any $0 \leq \sqrt{1/4-\lambda}$ the solution u in (5.2.31) belongs to $H^{1+s}(0,1)$ for any $s < \sqrt{1/4-\lambda}$. Otherwise, if $\sqrt{1/4-\lambda} > 1$ then u in (5.2.31) belongs to $L^2(0,1)$

Remark 5.2.1. *The previous example does not confirm a possible optimal regularity for the result of item* (2) *in Theorem 5.2.3. This suggests to address the following open question: Is there any value λ for which the solution of* (5.2.1) *belongs to $H^2(0,1)$ for any $f \in L^2(0,1)$. If yes, which is the range of λ's for which this occurs ?*

Proof of Theorem 5.2.3

By (5.2.9), (5.2.12) and using the fact that u satisfies the equation of problem (5.2.1), we have

$$\begin{aligned} |u''(r)| &\lesssim |f(r)| + \frac{\lambda}{r^2}(|u_S| + |u_R|) \\ &\lesssim |f(r)| + r^{-\frac{1}{2}} + r^{-\frac{3}{2}+\sqrt{1/4-\lambda}} \end{aligned} \tag{5.2.32}$$

Next, we split the proof in three steps when discussing the range of parameters λ as follows.

1. **The case $\sqrt{1/4-\lambda} \geq 1$.** By (5.2.32) we obtain $|u''(r)| \lesssim |f(r)| + r^{-\frac{1}{2}}$, and therefore we get that
$$u \in W^{2,p}(0,1), \quad \forall \quad 1 \leq p < 2.$$
Using the Sobolev embedding (5.2.30) we have
$$W^{2,p}(0,1) \hookrightarrow H^{1+s}(0,1), \qquad \text{for } s = \frac{3}{2} - \frac{1}{p} \tag{5.2.33}$$
From above, we find that
$$u \in H^{1+s}(0,1), \quad \forall \quad s < 1. \tag{5.2.34}$$

2. **The case $\frac{1}{2} \leq \sqrt{1/4-\lambda} < 1$.** Again, from (5.2.32) we have $|u''(r)| \lesssim |f(r)| + r^{-\frac{3}{2}+\sqrt{1/4-\lambda}}$, and in consequence
$$u \in W^{2,p}(0,1), \quad p < \frac{1}{3/2 - \sqrt{1/4-\lambda}}.$$
The continuous embedding (5.2.33) yields to
$$u \in H^{1+s}(0,1), \quad s < \sqrt{1/4-\lambda}. \tag{5.2.35}$$

3. **The case $0 < \sqrt{1/4-\lambda} < \frac{1}{2}$.** We have
$$\begin{aligned} |u'(r)| \lesssim & \quad r^{-\frac{1}{2}+\sqrt{1/4-\lambda}} + G_1(r) r^{-\frac{1}{2}+\sqrt{1/4-\lambda}} + (G_2(r) - G_2(0)) r^{-\frac{1}{2}-\sqrt{1/4-\lambda}} \\ & := I_1(r) + I_2(r) + I_3(r). \end{aligned} \tag{5.2.36}$$
In the sequel we estimate the terms I_1, I_2, I_3 in (5.2.36). By [97] we know that
$$r^\alpha \in H^m(0,1), \quad \forall \quad m < \alpha + \frac{1}{2},$$
and in consequence we get easily that
$$I_1 \in H^s(0,1), \quad s < \sqrt{1/4-\lambda}.$$
Then
$$I_3'(r) \lesssim |f(r)| + r^{-\frac{1}{2}},$$
therefore $I_3 \in W^{1,p}(0,1)$, for any $p < 2$. Using the continuous Sobolev embedding $W^{1,p}(0,1) \hookrightarrow H^s(0,1)$ for any $s < 1$, we obtain
$$I_3 \in H^s(0,1), \quad \forall \quad s < 1.$$

Let us now to study the regularity of the worst term in the expression of u', that is I_2.

Now we are looking for $0 < s < 1$ such that $I_2 \in H^s(0,1)$ or equivalent the quantities
$$A := \int_0^1 \int_0^1 \frac{\left|G_1(r_1) r_1^{-\frac{1}{2}+\sqrt{1/4-\lambda}} - G_1(r_2) r_2^{-\frac{1}{2}+\sqrt{1/4-\lambda}}\right|^2}{|r_1 - r_2|^{1+2s}} dr_1 dr_2,$$

must be finite. Note that we can obtain

$$A \lesssim \int_0^1 \int_0^1 \frac{|G_1(r_1)|^2 \big| r_1^{-\frac{1}{2}+\sqrt{1/4-\lambda}} - r_2^{-\frac{1}{2}+\sqrt{1/4-\lambda}} \big|^2 + r_2^{-1+2\sqrt{1/4-\lambda}} |G_1(r_2) - G_1(r_1)|^2}{|r_1 - r_2|^{1+2s}} dr_1 dr_2. \tag{5.2.37}$$

Taking into account that

$$\begin{aligned} |G_1(r)| &\lesssim (1 + r^{1-\sqrt{1/4-\lambda}}) \\ |G_1(r_1) - G_1(r_2)|^2 &\lesssim \big| r_1^{2-2\sqrt{1/4-\lambda}} - r_2^{2-2\sqrt{1/4-\lambda}} \big| \\ |G_2(r_1) - G_2(r_2)|^2 &\lesssim \big| r_1^{2+2\sqrt{1/4-\lambda}} - r_2^{2+2\sqrt{1/4-\lambda}} \big|, \end{aligned} \tag{5.2.38}$$

we have

$$\begin{aligned} A \lesssim & \int_0^1 \int_0^1 \frac{\big(1 + r_1^{2-2\sqrt{1/4-\lambda}}\big) \big| r_1^{-\frac{1}{2}+\sqrt{1/4-\lambda}} - r_2^{-\frac{1}{2}+\sqrt{1/4-\lambda}} \big|^2}{|r_1 - r_2|^{1+2s}} dr_1 dr_2 + \\ & + \int_0^1 \int_0^1 \frac{r_2^{-1+2\sqrt{1/4-\lambda}} \big| r_1^{2-2\sqrt{1/4-\lambda}} - r_2^{2-2\sqrt{1/4-\lambda}} \big|^2}{|r_1 - r_2|^{1+2s}} dr_1 dr_2 := A_1 + A_2. \end{aligned}$$

Then

$$A_1 \lesssim \int_0^1 \int_0^{r_1} \ldots + \int_0^1 \int_{r_1}^1 \ldots =: A_{11} + A_{12}.$$

With a change of variable we obtain

$$A_{11} \lesssim \int_0^1 (r_1^{1-2s} + r_1^{2\sqrt{1/4-\lambda}-1-2s}) dr_1 \int_{\mu=0}^1 \frac{\big|1 - \mu^{-\frac{1}{2}+\sqrt{1/4-\lambda}}\big|^2}{|1-\mu|^{1+2s}} d\mu$$

For all $\alpha \in \mathbb{R}$ there exists the constants C_1, C_2 depending on α such that

$$C_1 |1 - \mu| \le |1 - \mu^\alpha| \le C_2 |1 - \mu|, \quad \forall \mu \in [0, 2]. \tag{5.2.39}$$

According to this we have

$$A_{11} \lesssim \int_0^1 (r_1^{1-2s} + r_1^{2\sqrt{1/4-\lambda}-1-2s}) dr_1 \int_{\mu=0}^1 (1-\mu)^{1-2s} d\mu,$$

which is finite if $s < 1, \quad s < \sqrt{1/4 - \lambda}$. For A_{12} we have

$$\begin{aligned} A_{12} &\lesssim \int_0^1 \int_{\mu=1}^{\frac{1}{r_1}} (r_1^{1-2s} + r_1^{2\sqrt{1/4-\lambda}-1-2s}) \frac{|1 - \mu^{-\frac{1}{2}+\sqrt{1/4-\lambda}}|^2}{|1-\mu|^{1+2s}} d\mu dr_1 = \\ &= \int_0^{\frac{1}{2}} \int_{\mu=1}^2 \ldots + \int_0^{\frac{1}{2}} \int_{\mu=2}^{\frac{1}{r_1}} \ldots + \int_{\frac{1}{2}}^1 \int_{\mu=1}^{\frac{1}{r_1}} \ldots =: A_{12}^1 + A_{12}^2 + A_{12}^3. \end{aligned}$$

Then applying again (5.2.39) we obtain

$$A_{12}^3 \lesssim \int_{\mu=1}^2 (\mu - 1)^{1-2s} d\mu,$$

which converges if $s < 1$, respectively

$$A_{12}^1 \lesssim \int_0^{\frac{1}{2}} (r_1^{1-2s} + r_1^{2\sqrt{1/4-\lambda}-1-2s}) dr_1 \int_{\mu=1}^{2} (\mu-1)^{1-2s} d\mu < \infty,$$

iff $s < 1$, $s < \sqrt{1/4-\lambda}$. If $\sqrt{1/4-\lambda} < \frac{1}{2}$ then

$$\int_{\mu=2}^{\frac{1}{r_1}} \frac{|1-\mu^{-\frac{1}{2}+\sqrt{1/4-\lambda}}|^2}{|1-\mu|^{1+2s}} d\mu \leq C \int_2^{\infty} \frac{1}{\mu^{1+2s}} d\mu.$$

Therefore, A_{12}^2 is finite if $s > 0$. In consequence we obtain

$$A_1 \text{ is finite if } 0 < s < \sqrt{1/4-\lambda}.$$

Let us switch to A_2. Firstly, we have

$$A_2 = \int_0^1 \int_0^{r_1} \ldots + \int_0^1 \int_{r_1}^1 \ldots =: A_{21} + A_{22}.$$

With a change of variables we get

$$A_{21} = \int_0^1 r_1^{1-2s} dr_1 \int_{\mu=0}^1 \frac{\mu^{-1+2\sqrt{1/4-\lambda}} |1-\mu^{2-2\sqrt{1/4-\lambda}}|}{|1-\mu|^{1+2s}} d\mu$$

Splitting the integral with respect to μ in two parts we obtain

$$A_{21} \lesssim \int_0^1 r_1^{1-2s} dr_1 \int_{\mu=0}^{\frac{1}{2}} \mu^{-1+2\sqrt{1/4-\lambda}} d\mu + \int_0^1 r_1^{1-2s} dr_1 \int_{\mu=\frac{1}{2}}^1 \frac{1}{(1-\mu)^{2s}} d\mu,$$

which is finite if $s < \frac{1}{2}$.
Then A_{22} can be written as

$$\begin{aligned} A_{22} &= \int_0^1 r_1^{1-2s} \int_{\mu=1}^{\frac{1}{r_1}} \mu^{-1+2\sqrt{1/4-\lambda}} \frac{|1-\mu^{2-2\sqrt{1/4-\lambda}}|}{|1-\mu|^{1+2s}} d\mu dr_1 \\ &= \int_0^{\frac{1}{2}} \int_1^2 \ldots + \int_0^{\frac{1}{2}} \int_2^{\frac{1}{r_1}} \ldots + \int_{\frac{1}{2}}^1 \int_1^{\frac{1}{r_1}} =: A_{22}^1 + A_{22}^2 + A_{22}^3. \end{aligned}$$

Then

$$A_{22}^1 \lesssim \int_0^{\frac{1}{2}} r_1^{1-2s} dr_1 \int_1^2 \frac{1}{(\mu-1)^{2s}} d\mu$$

which is finite if $s < \frac{1}{2}$. Then

$$\begin{aligned} \int_{\mu=2}^{\frac{1}{r_1}} \mu^{-1+2\sqrt{1/4-\lambda}} \frac{|1-\mu^{2-2\sqrt{1/4-\lambda}}|}{|1-\mu|^{1+2s}} d\mu &\leq \int_{\mu=2}^{\frac{1}{r_1}} \frac{\mu^{-1+2\sqrt{1/4-\lambda}} \mu^{2-\sqrt{1/4-\lambda}}}{\mu^{1+2s}} d\mu \\ &= \int_{\mu=2}^{\frac{1}{r_1}} \mu^{-2s} d\mu \lesssim r_1^{-1+2s} + 1. \end{aligned}$$

Therefore,

$$A_{22}^2 \lesssim \int_0^{\frac{1}{2}} r_1^{1-2s}(r_1^{-1+2s}+1)dr_1,$$

which converges for any $s<1$. We finish to say that

$$A_2 \text{ is finite iff } 0<s<\frac{1}{2} \tag{5.2.40}$$

and consequently

$$u\in H^{1+s}(0,1), \quad \forall \quad s<\sqrt{1/4-\lambda}.$$

With this Theorem 5.2.1 is proved.

□

5.3. The multi-dimensional case

We consider the singular elliptic problem (5.1.1) for some particular configurations of Ω and f. Firstly, we study the regularity of the solutions when Ω is a ball centered at origin end f is a radial function. Secondly, in the general case where both Ω and f are not symmetric with respect to the origin, we determine the asymptotic behavior of the radial part of the solution.

5.3.1. The radial case

Let $\Omega=B_1(0)$ be the unit ball in $\mathbb{R}^N$ centered at origin. Assume $f\in L^2(B_1(0))$ is a radial function i.e. $f(x)=f(|x|)$. Then problem (5.1.1) reduces to

$$\begin{cases} -\Delta u-\lambda\frac{u}{|x|^2}=f(|x|), & x\in B_1(0),\\ u=0, & \partial B_1(0),\end{cases} \tag{5.3.1}$$

Next we will show that problem (5.3.1) reduces to the 1-d case presented in Section 5.2.

Explicit formulas using one-dimensional reduction

Let us introduce the functions

$$H_1(r):=\int_1^r t^{\frac{N}{2}-\sqrt{\lambda_\star-\lambda}}f(t)dt, \tag{5.3.2}$$

$$H_2(r):=\int_1^r t^{\frac{N}{2}+\sqrt{\lambda_\star-\lambda}}f(t)dt, \tag{5.3.3}$$

which are well defined for any $r>0$, respectively $r\geq 0$.

We claim

Proposition 5.3.1 (Subcritical case)**.** *Assume $\lambda < \lambda_\star$. Then the radial function*

$$u(r) = \frac{1}{2\sqrt{\lambda_\star - \lambda}}(H_2(0) - H_1(r))r^{\frac{2-N}{2}+\sqrt{\lambda_\star-\lambda}} + \frac{1}{2\sqrt{\lambda_\star - \lambda}}(H_2(r) - H_2(0))r^{\frac{2-N}{2}-\sqrt{\lambda_\star-\lambda}} \tag{5.3.4}$$

is the unique solution in $H_0^1(B_1(0))$ of problem (5.3.1)*.*

Proposition 5.3.2 (Critical case)**.** *Assume $\lambda = \lambda_\star$ then the radial function*

$$u(r) = \Big(\int_1^r t^{N/2}(\log t)f(t)dt\Big)r^{\frac{2-N}{2}} - \Big(\int_1^r t^{N/2}f(t)dt\Big)r^{\frac{2-N}{2}}\log r, \tag{5.3.5}$$

is the unique solution in $H_{\lambda_\star}$ of problem (5.3.1)*.*

Proof. The proofs of Proposition 5.3.1, 5.3.2 follow the same steps.

Firstly, we note that problem (5.3.1) is written in polar coordinates as follows

$$\begin{cases} -u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta_{S^{N-1}}u - \frac{\lambda}{r^2}u = f, & x \in B_1(0), \\ u'(0,\sigma) = u(1,\sigma), & \sigma \in \partial B_1(0), \end{cases} \tag{5.3.6}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on the N-d unit sphere. With the change of variables

$$u(r,\sigma) = r^\alpha v(r,\sigma),$$

we obtain (at least formally) the equation for v:

$$\begin{cases} -v_{rr} + (-2\alpha - (N-1))\frac{v_r}{r} + \frac{-\alpha(\alpha-1)-\alpha(N-1)-\lambda}{r^2}v - \frac{1}{r^2}\Delta_{S^{N-1}}v = f(r)r^{-\alpha}, & (r\sigma) \in B_1(0), \\ v(0,\sigma) = v(1,\sigma), & \sigma \in \partial B_1(0), \end{cases} \tag{5.3.7}$$

Next we choose $\alpha = (1-N)/2$ which verifies $-2\alpha-(N-1) = 0$ and therefore the equation (5.3.7) reduces to

$$\begin{cases} -v_{rr} - \frac{1}{r^2}\Delta_{S^{N-1}}v - \frac{\left(\lambda+\frac{1}{4}-\lambda_\star\right)}{r^2}v = f(r)r^{\frac{N-1}{2}}, & (r,\sigma) \in (0,1) \times S^{N-1}, \\ v(0,\sigma) = v(1,\sigma) = 0, & \sigma \in \partial B_1(0), \end{cases} \tag{5.3.8}$$

It suffices to solve the problem (5.3.7). More precisely, we are looking for radial solutions $v(r,\sigma) = v(r)$, and therefore the problem to be solved reduces to one dimensional problem

$$\begin{cases} -v_{rr} - \frac{\left(\lambda+\frac{1}{4}-\lambda_\star\right)}{r^2}v = f(r)r^{\frac{N-1}{2}}, & r \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{5.3.9}$$

that has been solved in Section 5.2. Applying the formulas in Theorems 5.2.1, 5.2.2 to problem (5.3.9), we conclude as follows.

For $\lambda < \lambda_\star$, we obtain

$$v(r) = \frac{1}{2\sqrt{\lambda_\star - \lambda}}(H_2(0) - H_1(r))r^{\frac{1}{2}+\sqrt{\lambda_\star-\lambda}} + (H_2(r) - H_2(0))r^{\frac{1}{2}-\sqrt{\lambda_\star-\lambda}}, \tag{5.3.10}$$

For $\lambda = \lambda_\star$ we get

$$v(r) = \Big(\int_1^r t^{N/2}(\log t) f(t)dt\Big) r^{1/2} - \Big(\int_1^r t^{N/2} f(t)dt\Big) r^{1/2}\log r. \tag{5.3.11}$$

Undoing the variables we finally get the conclusions of Propositions 5.3.1, 5.3.2.

□

Next we claim

Proposition 5.3.3. *(Regularity multi-dimensional) Assume $\lambda < \lambda_\star$, and u is the unique radial solution the solution in H_λ of problem* (5.3.1)*. It holds that*

1. *If $\sqrt{\lambda_\star - \lambda} \le \frac{N+1}{2}$ then*
$$u \in H^{1+s}(B_1(0)), \quad \forall s < \sqrt{\lambda_\star - \lambda}. \tag{5.3.12}$$
2. *If $\sqrt{\lambda_\star - \lambda} > \frac{N+1}{2}$ then*
$$u \in H^2(B_1(0)). \tag{5.3.13}$$

Proof. As in the proof of Theorem 5.2.3, by applying the Cauchy-Schwartz inequality we get that
$$|H_1(r)| \lesssim (r^{\frac{N+1}{2} - \sqrt{\lambda_\star - \lambda}} + 1) \tag{5.3.14}$$
respectively
$$|H_2(0)| \lesssim 1, \quad |H_2(r) - H_2(0)| \lesssim r^{\frac{N+1}{2} + \sqrt{\lambda_\star - \lambda}} \tag{5.3.15}$$
According to (5.3.14) and (5.3.15) we obtain

$$\frac{|u|}{r^2} \lesssim r^{-\frac{1}{2}} + r^{-\frac{N+2}{2} + \sqrt{\lambda_\star - \lambda}} + |f(r)|, \text{ a.e. in } (0,1). \tag{5.3.16}$$

If $\sqrt{\lambda_\star - \lambda} > \frac{N+1}{2}$ then we remark that $r^{-\frac{1}{2}}, r^{-\frac{N+2}{2} + \sqrt{\lambda_\star - \lambda}}, f(r) \in L^2(0,1)$. and therefore $u/r^2 \in L^2(0,1)$ which yields to $u \in H^2(B_1(0))$.

Otherwise, if $\sqrt{\lambda_\star - \lambda} \le \frac{N+1}{2}$, we obtain

$$\frac{u}{r^2} \in L^p(0,1), \quad 1 \le \frac{N}{\frac{N+2}{2} - \sqrt{\lambda_\star - \lambda}}.$$

and for such p we have $|D^2 u| \in L^p$ and therefore $u \in W^{2,p}(B_1(0))$.

Using the Sobolev embedding (5.2.30), we obtain that $u \in H^{1+s}(B_1(0))$ for all $s \le \sqrt{\lambda_\star - \lambda}$ and the proof finishes.

□

5.3.2. The non-radial case. Asymptotic behavior

The main result of this section is stated as follows.

Theorem 5.3.1. *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded open domain such that $0 \in \Omega$. Let us fix $\lambda \leq \lambda_\star$ and consider $v \in D(A_\lambda)$. Then the limit*

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon^{N-1}} \frac{v^2(\sigma)}{\varepsilon^{1+\sqrt{\lambda_\star - \lambda}}} d\sigma = l, \tag{5.3.17}$$

exists and is finite. In other words, if we define the radial part of v by

$$\overline{v}(r) = \int_{S^{N-1}} v(r\sigma) d\sigma,$$

it implies that

$$\exists \lim_{\varepsilon \to 0} \frac{|\overline{v}(\varepsilon)|}{\varepsilon^{-(N-2)/2+\sqrt{\lambda_\star - \lambda}}} = \sqrt{l} < \infty. \tag{5.3.18}$$

For the proof of Theorem 5.3.1 we need to use next lemma whose proof is given at the end of this section.

Lemma 5.3.1. *Assume J_ν is the Bessel function of order ν, $\nu \in \mathbb{R}$, $\nu \geq 0$, $\gamma_{n,\nu}$:=the nth positive zero of the Bessel function J_ν, $c_j := j(j+N-2)$, $j \geq 0$, and $m_j = \sqrt{c_j + \lambda_\star - \lambda}$. Then it holds that the series*

$$S := \sum_{j \geq 0} \sum_{n \geq 1} \frac{1}{\gamma_{n,m_j}^{4+2\sqrt{\lambda_\star - \lambda}} \int_0^1 t J_{m_j}^2(\gamma_{n,m_j} t) dt} < \infty, \tag{5.3.19}$$

converges.

Proof of Theorem 5.3.1

Without losing the generality we may assume $\Omega = B_1(0)$. Next, we consider the eigenvalue problem

$$\begin{cases} A_\lambda v = \mu v, & x \in B_1(0), \\ v = 0, & x \in S^{N-1}, \\ v \in H_\lambda, \end{cases} \tag{5.3.20}$$

where B_1 is the unit ball. The variational formulation of (5.3.20) is : find $\mu \in \mathbb{R}$ such that

$$\begin{cases} \int_{B_1(0)} (\nabla v + \frac{N-2}{2} \frac{x}{|x|^2} v)(\nabla \phi + \frac{N-2}{2} \frac{x}{|x|^2} \phi) + (\lambda_\star - \lambda) \frac{v\phi}{|x|^2} dx = \mu \int_{B_1} v\phi, \quad \forall \phi \in C_0^\infty(B_1(0)), \\ v \in H_\lambda, \quad v \neq 0. \end{cases} \tag{5.3.21}$$

arguments this is equivalent to

$$\begin{cases} \int_{B_1(0)} (\nabla v + \frac{N-2}{2} \frac{x}{|x|^2} v)(\nabla \phi + \frac{N-2}{2} \frac{x}{|x|^2} \phi) + (\lambda_\star - \lambda) \frac{v\phi}{|x|^2} dx = \mu \int_{B_1} v\phi, \quad \forall \phi \in H_\lambda, \\ v \in H_\lambda, \quad v \neq 0. \end{cases} \tag{5.3.22}$$

Next, we are going to solve the eigenvalue problem (5.3.20) by determining the family of eigenvalues $(\mu_n)_n$ and the corresponding eigenfunctions $(\phi_n)_n$ which will consist an orthogonal basis of $L^2(B_1(0))$. Due to the compactness embedding $H_\lambda \hookrightarrow L^2(B_1)$ the spectrum of the operator $A = -\Delta - \lambda/|x|^2$ is discrete, therefore, it makes sense to count the eigenvalues $(\mu_n)_n$. Using separation of variables, we are going to look for eigenfunctions of the form

$$v(x) = v(r\sigma) = v(r,\sigma) = v(r)f_j(\sigma),$$

where f_j are eigenfunctions for the Laplace-Beltrami operator on the sphere S^{N-1}. We know that $c_j = j(N+j-2)$, with $j \geq 0$, form the family of eigenvalues for the Laplace-Beltrami operator on the sphere. Due to that, the equation of (5.3.20) becomes

$$-v_{rr} - \frac{N-1}{r}v_r + \frac{c_j - \lambda}{r^2}v = \mu v, \quad j \geq 0. \tag{5.3.23}$$

and the compatibility boundary condition $v(1) = 0$. In the sequel we consider the change of variables $v(r) = r^{-(N-2)/2}u$. Then, the new variable u satisfies

$$\begin{cases} u_{rr} + \frac{u_r}{r} + \big(\mu - \frac{m_j^2}{r^2}\big)u = 0, & r \in (0,1], \quad j \geq 0, \\ u(1) = 0. \end{cases} \tag{5.3.24}$$

The general solution of problem (5.3.24) is given by

$$\begin{cases} u = AJ_{m_j}(\sqrt{\mu}r) + BY_{m_j}(\sqrt{\mu}r), & r \in (0,1], \quad j \geq 0, \\ u(1) = 0, \end{cases} \tag{5.3.25}$$

where by J_n we understand the nth Bessel function and respectively $Y_n = \int \frac{dx}{xJ_n^2(x)}$ denote the so-called Weber function (It is well known that J_n and Y_n are linear independent). Undoing the variables we note that v satisfies

$$\begin{cases} v(r) = r^{-(N-2)/2}\big(AJ_{m_j}(\sqrt{\mu}r) + BY_{m_j}(\sqrt{\mu}r)\big), & r \in (0,1], \quad j \geq 0, \\ v(1) = 0, \\ v \in H_\lambda. \end{cases} \tag{5.3.26}$$

If $j = 0$, i.e. $c_j = 0$, then $J_0 \sim C$ and $Y_0 \sim \log r$ as $r \to 0$. Hence $v \sim r^{-(n-2)/2}\log r$ which does not belong to H_λ unless $B = 0$.

If $j \geq 1$, then $c_j > 0$ and $J_{m_j}(r) \sim r^{m_j}$, $Y_{m_j}(r) \sim r^{-m_j}$ as $r \to 0$. Hence $v \sim r^{-(N-2)/2-m_j}$ which does not belong to H_λ unless $B = 0$. In conclusion, B must be equal to zero. Therefore, we get

$$\begin{cases} v = Ar^{-(N-2)/2}J_{m_j}(\sqrt{\mu}r), & x \in (0,1], \\ v(1) = AJ_{m_j}(\sqrt{\mu}) = 0, & \text{(the equation for } \mu\text{'s)}, \end{cases} \tag{5.3.27}$$

Hence the family of pairs $(\mu_{n,j}, \phi_{n,j})_{j\geq 0, n\geq 1}$ that solve (5.3.20) is given by

$$\begin{cases} \mu_{n,j} = \gamma_{n,m_j}^2, \\ \phi_{n,j} = r^{-(N-2)/2}J_{m_j}(\gamma_{n,m_j}r)f_j(\sigma), \end{cases} \tag{5.3.28}$$

where γ_{n,m_j} represents the n positive zero of Bessel functions J_{m_j}. The family $\{\phi_{n,j}\}_{n,j}$ above represents an orthogonal basis of $L^2(B_1(0))$. Without losing the generality, we may assume that

$$\int_{S^{N-1}} f_j^2(\sigma)d\sigma = 1, \quad \forall j \geq 1.$$

Next we consider the expansion in Fourier series of v that is

$$v = \sum_{j\geq 0, n\geq 1} a_{n,j}\phi_{n,j} = \sum_{j\geq 0, n\geq 1} a_{n,j} r^{-(N-2)/2} J_{m_j}(\gamma_{n,m_j} r) f_j(\sigma).$$

Then one can prove that

$$A_\lambda v = \sum_{n,j} a_{n,j} A_\lambda \phi_{n,j}.$$

Using the Plancherel formula, since $v \in D(A_\lambda)$ we obtain

$$\Big|\Big| A_\lambda v \Big|\Big|^2_{L^2(B_1(0))} = \sum_{n,j} a^2_{n,j}\gamma^4_{n,m_j} ||\phi_{n,j}||^2_{L^2(B_1(0))} < \infty. \tag{5.3.29}$$

But

$$||\phi_{n,j}||^2_{L^2(B_1(0))} = \int_0^1 t J^2_{m_j}(\gamma_{n,m_j} t)dt,$$

and according to (5.3.29) we have

$$\sum_{n,j} a^2_{n,j}\gamma^4_{n,m_j} \int_0^1 t J^2_{m_j}(\gamma_{n,m_j} t)dt < \infty. \tag{5.3.30}$$

According to Lemma 5.3.1 and (5.3.30), combined with Caucchy-Schwartz inequality we obtain

$$\sum_{j=0}^{\infty} \Big(\sum_{n=1}^{\infty} a_{n,j}\gamma_{n,m_j}^{-\sqrt{\lambda_\star - \lambda}} \Big)^2 < \infty. \tag{5.3.31}$$

Now, we are able finish the proof. Indeed, using the orthogonality of $\{f_j\}_j$ we obtain

$$\int_{S^{N-1}_\varepsilon} \frac{v^2}{\varepsilon^{1+2\sqrt{\lambda_\star - \lambda}}} d\sigma = \sum_j \Big(\sum_n a_{n,j} \frac{J_{m_j}(\gamma_{n,m_j}\varepsilon)}{\varepsilon^{\sqrt{\lambda_\star-\lambda}}} \Big)^2, \tag{5.3.32}$$

Because, $J_{m_j}(\gamma_{n,m_j}\varepsilon)/(\gamma_{n,m_j}\varepsilon)^{\sqrt{\lambda_\star - \lambda}}$ are uniformly bounded, the above series converges uniformly in ε due to (5.3.31). Finally, using the Weierstrass's uniform convergence theorem, we obtain

$$\lim_{\varepsilon\to 0} \int_{S^{N-1}_\varepsilon} \frac{v^2}{\varepsilon} d\sigma < \infty.$$

This finishes the proof of our Theorem.

Useful properties of Bessel functions-Proof of Lemma 5.3.1

In order to prove Lemma 5.3.1 we need to show some monotonicity properties involving the Bessel functions. We claim

Proposition 5.3.4. *The function*

$$\nu \mapsto \nu J_\nu(\nu/2)$$

is increasing.

Proposition 5.3.5. *For any $\nu > 0$ the function J_ν is strictly increasing on the interval $[0, \nu)$.*

Next we apply Propositions 5.3.4-5.3.5 to show Lemma 5.3.1 note.

Proof of Lemma 5.3.1. From Proposition 5.3.4 we remark that there exists a positive constant C (independent of ν) such that

$$J_\nu(\nu/2) \geq \frac{C}{\nu}, \quad \forall \nu \geq 0. \tag{5.3.33}$$

In order to facilitate the computations next we will write " $\gtrsim 1$" and " $\lesssim 1$" instead of " $\geq C$" respectively " $\leq C$".

Note that the series S in (5.3.19) may be written as

$$S = \sum_{j \geq 0} \sum_{n \geq 1} \frac{1}{\gamma_{n,m_j}^{2+2\sqrt{\lambda_\star - \lambda}} \int_0^{\gamma_{n,m_j}} t J_{m_j}^2(t) dt}.$$

Now we are able to give lower bounds for the quantity $\int_0^{\gamma_{n,m_j}} t J_{m_j}^2(t) dt$ in order to assure the convergence of S. Indeed, by Proposition 5.3.5 we get successively

$$\begin{aligned} \int_0^{\gamma_{n,m_j}} t J_{m_j}^2(t) &\geq \int_0^{\gamma_{1,m_j}} t J_{m_j}^2(t) dt \geq \int_{m_j/2}^{m_j} t J_{m_j}^2(t) dt \geq (m_j - \frac{m_j}{2}) \min_{t \in (m_j/2, m_j)} t J_{m_j}^2(t) \\ &= (m_j - \frac{m_j}{2}) c_j J_{m_j}^2(m_j/2) = \frac{m_j^2}{2} J_{m_j}^2(m_j/2). \end{aligned} \tag{5.3.34}$$

Here we have used that $\gamma_{1,m_j} > m_j$ (see for instance [52], [80]) Consequently, applying (5.3.33) we obtain

$$\int_0^{\gamma_{n,m_j}} t J_{m_j}^2(t) dt \gtrsim 1,$$

Hence,

$$S \leq \sum_{j \geq 0} \sum_{n \geq 1} \frac{1}{\gamma_{n,m_j}^2}.$$

On the other hand, (cf. [52], [80]) if $\nu > 1/2$ then

$$n\pi + \nu - 1/2 \leq \gamma_{n,\nu} \leq n\pi + \nu\pi/2 - \pi/4. \tag{5.3.35}$$

and in particular we have $\gamma_{n,m_j} \gtrsim n + m_j$. Using this we get from above that

$$\begin{aligned} S &\lesssim \sum_j \sum_n \frac{1}{(n + m_j)^{2+2\sqrt{\lambda_\star - \lambda}}} = \sum_j \Big(\frac{1}{(1 + m_j)^{2+2\sqrt{\lambda_\star - \lambda}}} + \frac{1}{(2 + m_j)^{2+2\sqrt{\lambda_\star - \lambda}}} + \dots \Big) \\ &\lesssim \sum_j \frac{1}{(1 + m_j)^{1+2\sqrt{\lambda_\star - \lambda}}} \lesssim \sum_j \frac{1}{j^{1+2\sqrt{\lambda_\star - \lambda}}} < \infty. \end{aligned} \tag{5.3.36}$$

With this the result of Lemma (5.3.1) is obtained for $\lambda < \lambda_\star$. For the critical case we can pass to the limit as $\lambda \to \lambda_\star$ using the fact that, as pointed out in Chapter 3, the solution of problem $A_\lambda u_\lambda = f$ converges in H_λ to the solution of the limit problem $A_{\lambda_\star} u_{\lambda_\star} = f$.

$\square$

Proof of Proposition 5.3.4. We proceed as in Watson [112], pp. 260-261. Firstly, (cf. [112], pp. 253) the function J_ν admits the integral representation

$$J_\nu(\nu x) = \frac{1}{\pi}\int_0^\pi e^{-\nu F(\theta,x)}d\theta, \quad 0 < x \leq 1, \tag{5.3.37}$$

where $F(\theta, x)$ is given by

$$F(\theta,x) = \log\frac{\theta + \sqrt{\theta^2 - x^2\sin^2\theta}}{x\sin\theta} - \sqrt{\theta^2 - x^2\sin^2\theta}\cot\theta.$$

Next, one can verify that

$$F_\theta(\theta,x) = \frac{(1-\theta\cot\theta)^2}{\sqrt{\theta^2 - x^2\sin^2\theta}} + \sqrt{\theta^2 - x^2\sin^2\theta}, \tag{5.3.38}$$

$$F(\theta,x) \geq 0. \tag{5.3.39}$$

$$\lim_{\theta\to 0} F(\theta,x) = \log\big(\frac{1}{x} + \frac{1}{x}\sqrt{1-x^2}\big), \quad \lim_{\theta\to\pi} F(\theta,x) = \infty \tag{5.3.40}$$

In particular,

From above observe that

$$\begin{aligned}
\frac{d\{vJ_\nu(\nu/2)\}}{d\nu} &= \frac{1}{\pi}\int_0^\pi e^{-\nu F(\theta,1/2)}d\theta - \frac{\nu}{\pi}\int_0^1 F(\theta,1/2)e^{-\nu F(\theta,1/2)}d\theta \\
&= \frac{1}{\pi}\Big[\theta e^{-\nu F(\theta,1/2)}\Big]_0^\pi + \frac{\nu}{\pi}\int_0^\pi \{\theta F_\theta(\theta,1/2) - F(\theta,1/2)\}e^{-\nu F(\theta,1/2)}d\theta \\
&= \frac{\nu}{\pi}\int_0^\pi \{\theta F_\theta(\theta,1/2) - F(\theta,1/2)\}e^{-\nu F(\theta,1/2)}d\theta, \qquad (5.3.41)
\end{aligned}$$

since the integrated part vanishes at each limit due to (5.3.40). Next, we note that, $F(\cdot, 1/2)$ is a convex function in θ, i.e.,

$$F_{\theta\theta}(\theta,1/2) \geq 0.$$

Consequently,

$$\theta F_{\theta\theta}(\theta,1/2) \geq 0,$$

which is to say

$$\frac{d}{d\theta}\{\theta F(\theta,1/2) - F_\theta(\theta,1/2)\} \geq 0.$$

Integrating from 0 to θ we obtain

$$\theta F_\theta(\theta,1/2) - F(\theta,1/2) \geq 0. \tag{5.3.42}$$

It follows from (5.3.42) and (5.3.41) that

$$\frac{d\{vJ_\nu(\nu/2)\}}{d\nu} \geq 0,$$

and the proof of Proposition 5.3.4 is finished. $\square$

Proof of Proposition 5.3.5. We know that J_ν satisfies the differential equation

$$J_\nu^{''} + \frac{J_\nu^{'}}{t} + \Big(1 - \frac{\nu^2}{t^2}\Big) J_\nu = 0. \tag{5.3.43}$$

Next, let $x \leq \nu$ be fixed. Then we multiply (5.3.43) by t and we integrate from 0 to x. After computation we obtain

$$xJ_\nu^{'}(x) = \int_0^x \Big(\frac{\nu^2}{t} - t\Big) J_\nu dt,$$

which is positive due to election of x and positivity of J_ν on $(0, \nu)$ which is due to $\gamma_{1,\nu} > \nu$, see [52]). Therefore, $J_\nu'(x) > 0$ for any $x \leq \nu$, and the proof is complete. □

5.3.3. Appendix-overview on spectral properties

In this section we remind some basic spectral results that we have used in Section 5.3.2.

If $\lambda < \lambda_\star$ the operator A_λ generates an equivalent norm in $H_0^1(\Omega)$. Due to the Hardy inequality we have

$$(A_\lambda u, u) \geq \Big(1 - \frac{\lambda}{\lambda_\star}\Big) \int_\Omega |\nabla u|^2,$$

and so A_λ is coercive in $H_0^1(\Omega)$. Given $f \in L^2(\Omega)$, via Lax-Milgram Theorem, there exists a unique weak solution u of (1) which solves the variational formulation

$$\begin{cases} u \in H_0^1(\Omega) \\ \int_\Omega \nabla u \cdot \nabla v dx - \lambda \int_\Omega \frac{uv}{|x|^2} dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{5.3.44}$$

This tells that A_λ is an isomorphism from $H_0^1(\Omega)$ into $L^2(\Omega)$. We consider now the inverse operator

$$A_\lambda^{-1} = (-\Delta - \frac{\lambda}{|x|^2})^{-1} : L^2 \to H_0^1(\Omega) \hookrightarrow L^2.$$

Proposition 5.3.6. *The inverse operator A_λ^{-1} is self-adjoint and compact from $L^2(\Omega)$ to $L^2(\Omega)$.*

Proof of Proposition 6.3.1. 1. A_λ^{-1} is self-adjoint.

Let be $f, g \in L^2(\Omega)$ and set $u = A_\lambda^{-1} f$, $v = A_\lambda^{-1} g$. By the self-adjointness of A_λ we have

$$(A_\lambda^{-1} f, g) = (u, A_\lambda v) = (A_\lambda u, v) = (f, A_\lambda^{-1} g).$$

2. A_λ^{-1} is bounded.

Firstly we have,

$$C||u||^2_{H_0^1(\Omega)} \leq (A_\lambda u, u) = (f, u) \leq ||f||_{L^2(\Omega)} ||u||_{L^2(\Omega)}.$$

Therefore there exists a positive constant C such that

$$||u||_{H_0^1(\Omega)} \leq C||f||_{L^2(\Omega)}.$$

Then we obtain

$$||A_\lambda^{-1} f||_{L^2(\Omega)} = ||u||_{L^2(\Omega)} \leq C||u||_{H_0^1(\Omega)} \leq C||f||_{L^2(\Omega)}.$$

3. A_λ^{-1} is compact.
   This comes from the fact that a priori A_λ^{-1} is defined from $L^2(\Omega)$ to $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is compact embedded in $L^2(\Omega)$.

□

Lemma 6.3.1 and Theorem vi.11 (Brezis [21]) imply

Proposition 5.3.7 (see [111])**.** *There exists an orthonormal basis $\{e_k\}_{k\geq 1}$ of $L^2(\Omega)$ constituted by the eigenvalues of A_λ^{-1}. The sequence of the corresponding eigenvalues $\{\mu_k\}_{k\geq 1}$ is monotone decreasing to 0.*

If e is an eigenfunction of A_λ^{-1} and μ is the corresponding eigenvalue, then e is an eigenfunction of A_λ with the corresponding eigenvalue $1/\mu$. Therefore, from the previous Lemma we have

Proposition 5.3.8. *There exists an orthonormal basis $\{e_k\}_{k\geq 1}$ of $L^2(\Omega)$ constituted by the eigenvalues of A_λ. The sequence of the corresponding eigenvalues $\{\mu_k\}_{k\geq 1}$ is monotone increasing to infinity i.e.*

$$\begin{cases} -\Delta e_k - \frac{\lambda}{|x|^2} e_k = \mu_k e_k, & x \in \Omega, \\ e_k = 0, & x \in \partial\Omega. \end{cases} \tag{5.3.45}$$

and

$$0 < \mu_1 \leq \mu_2 \leq \ldots \to \infty.$$

Moreover, from (5.3.45) we obtain $e_k \in D(A_\lambda)$.

We conclude with the following theorem.

Proposition 5.3.9. *Let us consider e_k and μ_k from Lemma 5.3.8. Then $\{\frac{e_k}{\sqrt{\mu_k}}\}_{k\geq 1}$ is an ortonormal basis of $H_0^1(\Omega)$.*

In the same way, Propositions above can be extended in the critical case $\lambda = \lambda_\star$ to the space $H_{\lambda_\star}$ larger than $H_0^1(\Omega)$.

The computation of the spectrum of A_λ

The spectrum can be explicitly computed. This gives detailed information about the singularities of the solution of (1). Writing in spherical coordinates $x = (r, \sigma)$, $r \in (0, 1)$ and $\sigma \in S^{N-1}$ we denote by f_j the eigenfunctions of the Laplace-Beltrami operator which constitute an orthonormal basis of $L^2(S^{N-1})$. We call by c_j the corresponding eigenvalues which are given by $c_j = j(j + N - 2)$ with $j = 0, 1, 2....$ Then, we look for the eigenfunctions in separated variables of the form

$$e_j(x) = e_j(r, \sigma) = \phi(r) f_j(\sigma).$$

Using the representation of the Laplacian in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}}$$

e_j verifies the eigenvalue problem

$$\phi'' + \frac{N-1}{r}\phi' + \Big(\frac{\lambda - c_j}{r^2} + \mu\Big)\phi = 0.$$

where μ is the eigenvalue corresponding to e_j. Taking the change of variable

$$\psi(r) = r^{\frac{N-2}{2}}\phi(r)$$

we obtain the Bessel equation in the new variable:

$$\psi'' + \frac{\psi'}{r} + \Big(\mu - \frac{\lambda_\star + c_j - \lambda}{r^2}\Big)\psi = 0, \tag{5.3.46}$$

with the boundary conditions $\psi(1) = 0$ and $\psi'(0) = 0$ (hidden in the variational formulation). Now we set

$$m(\lambda, j) = \sqrt{\lambda_\star + c_j - \lambda}. \tag{5.3.47}$$

Hence, the solution of the equation (5.3.46) is (cf. Bowman [20])

$$\psi_j(r) = J_m(\sqrt{\mu}r),$$

where J_m is the m-th fractional Bessel function. Any Bessel function J_m has an infinity number of zeros for any $m > -1$. Imposing $\psi_j(1) = 1$ we find $J_m(\sqrt{\mu}) = 0$ and consequently $\mu_{j,n} = z_{m,n}^2$ where $z_{m,n}$ are the zeros of the Bessel function J_m. Undoing the change of variable we have

$$\phi(r) = \phi_{j,n}(r) = r^{-\frac{N-2}{2}} J_m(z_{m,n}r).$$

Therefore we can reorganize the information in the following lemma.

Proposition 5.3.10. *There exists an orthogonal basis of $L^2(\Omega)$ and $H_0^1(\Omega)$ constituted by the two-parameter family of functions*

$$e_{j,n} = r^{-\frac{N-2}{2}} J_m(z_{m,n}r) f_j(\sigma), \quad j \geq 0, n \geq 1,$$

which are the eigenfunctions of L_λ with the corresponding eigenvalues $\mu_{j,n} = z_{m,n}$. These eigenvalues are the zeros of the mth Bessel function J_m. The index $m = m(\lambda, j) > 0$ is defined in (5.3.47).

Chapter 6

Adaptive finite element approximations for the Schrödinger operator with singular potential

Abstract. We analyze the performance of FEM (based on first-order polynomials) for the elliptic problem associated to the singular operator $A_\lambda := -\Delta - \lambda/|x|^2$. When applying the standard FEM on regular meshes, we prove weaker convergence rates than occur for the standard Laplacian, due to the lack of regularity of A_λ. To recover the optimal complexity of FEM, we build non-heterogenous triangulations adapted to the singularity of the problem.

6.1. Introduction

During the last decades, there has been intensive work in numerical analysis, trying to develop efficient numerical algorithms for solving PDEs.

In this chapter we will focus in particular on the Finite Element Method (FEM) which is a very well-known and useful numerical method to approximate PDEs.

A finite element scheme allows to compute the numerical solutions by reducing the considered PDE to linear algebraic systems. The functional space in which the continuous solution stands is infinite dimensional whereas a FE approximation lies in a finite dimensional space. This is a very important issue in the numerical analysis since in general the solutions of the continuous problem cannot be computed explicitly. In view of that, the implementation of efficient numerical methods aims to identify and describe the properties of the continuous solutions which cannot be deduced from the theoretical analysis.

In this chapter we discuss the performance of finite element approximations based on piecewise linear elements (abv. P1), when solving the singular elliptic problem

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{6.1.1}$$

Depending on the location of the singularity $x = 0$ (i.e., inside Ω or on its boundary) we consider the range of parameters λ given by the Hardy inequalities presented in Chapter 2.

We will assume that $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$. More precisely, for $N = 1$ or $N = 2$ we analyze the case of boundary singularity, whereas for $N \geq 3$ we analyze the case of interior singularity.

In a generic sense, the description of the FE approximations depends on a parameter $h > 0$ meant to be small. More precisely, when the parameter h is decreasing to zero, the efficient numerical schemes provide better accuracy for the error and the corresponding numerical solution, denoted by u_h, converges to the continuous solution u in a functional setting that must be well specified.

In particular, in the case of the Laplacian, i.e., (6.1.1) with $\lambda = 0$, by means of first order polynomials (P1) and regular meshes of size h one can obtain a convergence rate of order $O(h)$ in the H_0^1-norm, in any dimension, provided the solution of the continuous problem possesses H^2 regularity, that is

$$||u - u_h||_{H_0^1(\Omega)} \lesssim h|D^2 u|_{L^2(\Omega)}. \tag{6.1.2}$$

This classical result has been extended to the case of higher order polynomials, see e.g. [43], where it was proved that the rate of convergence is proportional to the polynomial degree used in the finite element approximation.

In what follows we only consider the case of first order polynomials which ensures the convergence in (6.1.2). This order of convergence requires a number of degrees of freedom (DoF) of order $n(h) \sim h^{-N}$ (which represents the number of unknowns of the finite dimensional problem).

Due to the presence of the singularity, according to Chapter 5, the standard elliptic regularity fails for (6.1.1) and therefore, the classical FE method provides weaker convergence rates than for the Laplacian (which corresponds to $\lambda = 0$). This will be explained in details later on.

In Chapter 5 we proved that the regularity of the exact solution of (6.1.1) drops down from H^2 to a certain H^{1+s} for some constant $0 < s = s(\lambda) < 1$. Similar lack of regularity occurs for the solution of the Laplace equation in domains with cracks or corners (see e.g. [76]). In these cases, when applying standard FEM with uniform meshes, the convergence rate falls down from h to h^s. In consequence, the cost we need to obtain an error h^s is $n(h) = h^{-N}$. Of course, we can reach an error h but we have to refine appropriately the mesh. This means that the number of unknowns necessary to reach an error h is of order $n(h) \sim h^{-N/s}$, $s < 1$. Therefore, the lack of regularity makes the implementation of the numerical method costly and this is really inconvenient in practice.

The question that arises is: can we compensate the lack of regularity in the computational cost by using a non-standard finite element approximation?

In the recent past, various adaptations of FEM have been introduced in order to improve the convergence rates for problems with less regular solutions. Particularly, Ted Belytschko [16] introduced in 1999 an important extension of FEM called X-FEM (Extended Finite Element Method). X-FEM consists in adding to the basis of the classical finite elements a new function which is adapted to the singularity of problem under consideration (corners, cracks, etc...). In view of that, X-FEM yields an order of convergence h without having regularity H^2 for the solution. There is a large variety of papers regarding this aspect, mainly referring to Babuska [101], [7] and the references therein.

Another very important method which has been developed in the last decades is the so-called Adaptive FEM (or abv. AFEM). This numerical approximation is an iterative method in which the error is quantified and distributed on non-uniform meshes due to the singular features of the problem (see e.g. [32], [53], [91] and references therein). In various problems this method has an optimal computational cost.

The aim of this chapter consists in constructing FE element approaches for the solution $u \in H_\lambda$ of problem (6.1.1) in order to optimize the rate of convergence of FEM in the H_λ-norm.

In order to improve the computational cost, we build some extensions of the classical FE approximations (see for instance the pioneering works [49], [16], [91]). Hence, we improve the convergence rates by using heterogeneous meshes which are adapted to the singularity of the potential. For the one-dimensional problem, we use an adapted algorithm based on a priori estimates to show optimal convergence rates.

By means of an adaptive algorithm based on classical a posteriori techniques, we show some numerical experiments for the two-dimensional problem. We implement this algorithm and discuss the pathological behavior of the numerical solution and the adapted meshes of (6.1.1) with respect to the parameter λ. The numerical code we use is an adaptation of the tutorial code by [42] developed for the Laplacian in L-shaped domains or domains with cracks.

6.2. Standard FEM

6.2.1. One-dimensional case

We consider the one-dimensional problem (6.1.1) in the interval $\Omega = [0, 1]$ in the subcritical case $\lambda < \lambda_\star = 1/4$ for which $H_\lambda = H_0^1(0, 1)$. Let us also consider the sequence $(x_i)_{i=0,n+1}$, $n \in \mathbb{N}$, to be a division of the interval $(0 = x_0 < x_1 < \ldots < x_n < x_{N+1} = 1)$. For all $i \in \{1, \ldots, n\}$, we denote by $I_i = [x_i, x_{i+1}]$ of end points x_i and x_{i+1}. We introduce the following piecewise linear and continuous basis functions ϕ_i generating the P1 finite element space

$$\varphi_i(r) = \begin{cases} \frac{r-x_{i-1}}{x_i-x_{i-1}}, & r \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-r}{x_{i+1}-x_i}, & r \in [x_i, x_{i+1}], \\ 0, & rest. \end{cases} \tag{6.2.1}$$

The finite dimensional Hilbert space V_n, $V_n \subset H_0^1(0, 1)$ with, $\dim_{\mathbb{R}} V_n = n$, is generated by the functions $\varphi_1, ..., \varphi_n$, i.e.,

$$V_n =< \varphi_1, \varphi_2, ..., \varphi_n > .$$

By Lax-Milgram lemma, there exists a unique $u_n \in V_n$ which solves the variational problem

$$\begin{cases} u_n \in V_n \\ \int_0^1 u_n' v_n' dr - \lambda \int_0^1 \frac{u_n v_n}{r^2} dr = \int_0^1 f v_n dr, \quad \forall v_n \in V_n, \end{cases} \tag{6.2.2}$$

or equivalently

$$\begin{cases} u_n \in V_n \\ a(u_n, v_n) = (f, v_n), \quad \forall v_n \in V_n, \end{cases} \tag{6.2.3}$$

where $a(u,v) = \int_0^1 u'v'dr - \lambda \int_0^1 \frac{uv}{r^2}dr$ is a bilinear form, continuous and coercive. Then, up to a constant, u_n is the projection of u on V_n (Céa's Lemma) i.e.

$$||u - u_n||_{H_0^1(0,1)} \lesssim \inf_{v_n \in V_n} ||u - v_n||_{H_0^1(0,1)}. \tag{6.2.4}$$

In view of (6.2.4), in order to evaluate the difference $u - u_n$, we need to chose a convenient $v_n \in V_n$ depending on u to evaluate the difference $u - v_n$. The simplest case is when v_n is the interpolation of u on the nodes $(x_i)_{i=1,n}$, i.e

$$v_n(r) = \sum_{i=1}^{n} u(x_i)\phi_i(r), \quad \forall r \in (0,1). \tag{6.2.5}$$

Due to the Sobolev inclusion $H^1(0,1) \subset C(0,1)$, the solution u is continuous. Thus the values $u(x_i)$ are well defined and therefore the interpolation v_n is also well defined. Moreover v_n belongs to $H_0^1(0,1)$.

Now we are able to state our first approximation result.

Proposition 6.2.1. *Let us consider that $f \in L^2(0,1)$ and $u \in H^{1+s}(0,1)$ for some $0 < s = s(\lambda) < 1$, is the solution of problem (6.1.1). If v_n is the interpolation defined in (6.2.5) then*

$$||u' - v_n'||^2_{L^2(0,1)} \leq \sum_{i=0}^{n} (x_{i+1} - x_i)^{2s} |u|^2_{H^{1+s}(I_i)}, \tag{6.2.6}$$

where the semi-norm $|\cdot|_{H^{1+s}(0,1)}$ is given by

$$|u|^2_{H^{1+s}(0,1)} = \int_0^1 \int_0^1 \frac{|u'(t_1) - u'(t_2)|^2}{|t_1 - t_2|^{1+2s}} dt_1 dt_2 < \infty.$$

Proof of Proposition 6.2.1. Firstly we have

$$\begin{aligned} ||u' - v_n'||^2_{L^2(0,1)} &= \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} |u' - v_n'|^2 dr = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} \Big| u'(r) - \Big(\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \Big) \Big|^2 dr \\ &= \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2} \Big| \int_{x_i}^{x_{i+1}} \big(u'(r) - u'(t)\big) dt \Big|^2 dr. \end{aligned}$$

From the Cauchy-Schwartz inequality we obtain

$$|u' - v_n'|^2_{L^2(0,1)} \leq \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)} \int_{x_i}^{x_{i+1}} |u'(r) - u'(t)|^2 dt dr.$$

Using the definition of the fractional Sobolev semi-norm we obtain

$$\begin{aligned} ||u' - v_n'||^2_{L^2(0,1)} &\leq \sum_{i=0}^{n} \frac{1}{(x_{i+1} - x_i)} \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} \frac{|u'(r) - u'(t)|^2}{|r - t|^{1+2s}} (x_{i+1} - x_i)^{1+2s} dt dr \\ &= \sum_{i=0}^{n} (x_{i+1} - x_i)^{2s} \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} \frac{|u'(r) - u'(t)|^2}{|r - t|^{1+2s}} dt dr \\ &= \sum_{i=0}^{n} (x_{i+1} - x_i)^{2s} |u|^2_{H^{1+s}(0,1)}. \end{aligned}$$

Thus the estimation (6.2.6) holds true. $\square$

Uniform meshes. For any $n \in \mathbb{N}$ we introduce on the interval $[0,1]$ the classical uniform mesh of size $h = 1/(n+1)$, nodes $x_j = jh$ for any $j = 0, ..., n+1$. As a consequence of Proposition 6.2.1 we have

Corolary 6.2.1. *Let be $u \in H^{1+s}(0,1)$ for some $0 < s = s(\lambda) \leq 1$, the solution of (6.1.1) and u_h the solution of the discrete Galerkin approximation defined in (6.2.3). Then, there exists a constant $C > 0$ independent of h such that*

$$||u - u_h||_{H_0^1(0,1)} \leq Ch^s |u|_{H^{1+s}(0,1)}, \tag{6.2.7}$$

6.2.2. Multi-dimensional case

Triangulations.

In order to fix the ideas, for any dimension $N \geq 2$, let us consider a bounded and open domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ such that $0 \in \overline{\Omega}$. To do the explanation easier we consider $N \in \{2,3\}$. Technically speaking, the analysis for $N \geq 4$, is the same as for $N = 3$, but requires a more general terminology. Without losing generality, in what follows, we consider that Ω is a connected polygonal domain.

Let us choose $h > 0$ small enough and $\mathcal{T}_h$ to be a partition of Ω with the elements (say polyhedrons) $T_1, T_2, ..., T_n$. For $N \in \{2,3\}$ these elements correspond to triangles, respectively tetrahedrons. For the next definition we will use the terminology corresponding to $N = 2$.

Definition 6.2.1. *We say that $\mathcal{T}_h$ is a triangulation of Ω, if it a partition of Ω, i.e.*

$$\overline{\Omega} = \cup_{i=1}^n T_i$$

satisfying the conditions:

a) each element T_i has a non-empty interior;

b) $\overset{\circ}{T_i} \cap \overset{\circ}{T_j} = \emptyset$ for each distinct $T_i, T_j \in \mathcal{T}_h$, where $\overset{\circ}{T_i}$ denotes the interior of the triangle T_i.

c) if $F = \overset{\circ}{T_i} \cap \overset{\circ}{T_j} \neq \emptyset$ then F is a common side of T_i and T_j;

d) there exist two positive constants c_1, c_2 independent of h such that

$$c_1 h \leq h_T \leq c_2 h, \quad \forall T \in \mathcal{T}_h,$$

where h_T is the diameter of T.

Definition (6.2.1) can de extended to any dimension $N \geq 3$ with an appropriate terminology. The points $\{x_i\}_{i=1,m}$ are called the vertices of the polyhedrons T_i and represent the nodes of the triangulation. The above definition suggests us that the number of nodes x_i is comparable with the number of polyhedrons T_i, up to a constant independent of h. More precisely,

$$m \simeq O(n) = O\Big(\frac{1}{h^N}\Big).$$

We associate to each node x_i the pyramidal lineal function ϕ_i which is unique determined by knowing its value at any node of the triangulation:

$$\phi_i(x_j) = \delta_{i,j}, \quad \forall 1 \leq i, j \leq m,$$

where $\delta_{i,j}$ is the Kronecker's symbol (see Figure 6.1).

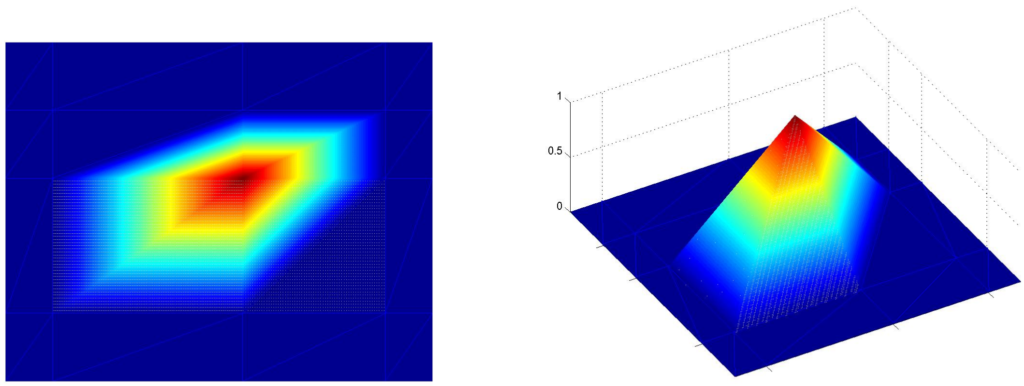

Figure 6.1: The piecewise linear basis function used in FEM

For a polyhedron $T \in \mathcal{T}_h$ we denote the diameter of T by h_T. We call also by ρ_T to be the radius of the biggest N-d ball that could be put inside of T. The reference polyhedron $\hat{\mathrm{T}}$ is the unit N-cube $[0,1]^N$.

Definition 6.2.2. *We say that a triangulation is regular if moreover we impose the constraint: there exists a uniform constant $c > 0$ such that*

$$\frac{h_T}{\rho_T} < c, \quad \forall T \in \mathcal{T}_h. \tag{6.2.8}$$

Comparing with d)., this is equivalent to the existence of two positive constants C_1, C_2 such that

$$C_1 h \leq \rho_T \leq C_2 h, \quad \forall T \in \mathcal{T}_h. \tag{6.2.9}$$

As we have seen in Chapter 5, the solution u of (6.1.1) belongs to $H^{1+s}(\Omega) \cap H_0^1(\Omega)$ for some $0 < s = s(\lambda) \leq 1$ provided $f \in L^2(\Omega)$.

Next we recall the notion of interpolation operators illustrated in Clément [43]. These operators produce a priori estimates for the error and are fundamental in the analysis of the convergence of the finite element approximations.

Interpolation operators

Continuity. If the solution u of (6.1.1) is continuous (which may occur in lower dimensions) we have the standard definition of the interpolation operator Π as follows: for any $T \in \mathcal{T}_h$,

$\Pi_T : H^{1+s}(T) \to H^1(T)$ is given by

$$\Pi_T u(x) = \sum_{x_i \in \overline{T}} u(x_i)\phi_i(x), \quad \forall x \in T. \tag{6.2.10}$$

Then

$$\Pi u(x) = \sum_{i}^{M} u(x_i)\phi_i(x), \quad \forall x \in \Omega.$$

One of the main advantages of Π above is that it maintains the values of u at each node, that is $\Pi(x_i) = u(x_i)$, for all $i = 1, \ldots, n$.

Lack of continuity. If the continuity of u fails then we cannot define the operator Π as above because we are not sure that the values $u(x_i)$ exist. Therefore we are obliged to introduce a more general interpolation operators.

Firstly, we denote by S_i the union of polyhedrons to whom x_i is vertex, i.e.,

$$S_i = \cup_{x_i \in T_j} T_j.$$

For a fixed index i we define p_i to be the unique polynomial of degree 1 (P1) satisfying

$$(u - p_i, p)_{L^2(S_i)} = 0, \quad \forall p \in P1.$$

Then we define the operator Π as follows: for any $T \in \mathcal{T}_h$, $\Pi_T^{\mathcal{C}} : H^{1+s}(T) \to H^1(T)$ is given by

$$\Pi_T^{\mathcal{C}} u(x) = \sum_{x_i \in T} p_i(x_i)\phi_i(x), \quad \forall x \in T, \tag{6.2.11}$$

and consequently,

$$\Pi^{\mathcal{C}} u(x) = \Pi_T^{\mathcal{C}} u(x), \quad \text{for } x \in \overline{T}.$$

The operator $\Pi_T^{\mathcal{C}}$ is called the Clement interpolation (see e.g. [43]).

Error estimates of fractional order. Let Π, $\Pi^{\mathcal{C}}$ be the interpolation operators above defined. We consider a domain Ω as above and a corresponding regular triangulation $\mathcal{T}_h$, $h > 0$. Given an arbitrary $s > 0$, we have the following approximation results:

Theorem 6.2.1. *Suppose $H^{1+s}(\Omega) \subset C(\overline{\Omega})$. Then there exists a constant $c_1 > 0$ independent of h such that, for any $u \in H^{1+s}(\Omega)$, the following estimate holds:*

$$|\nabla(u - \Pi u)|_{L^2(\Omega)} \leq c_1 h^s |u|_{H^{1+s}(\Omega)}. \tag{6.2.12}$$

Theorem 6.2.2. *Suppose $H^{1+s}(\Omega) \not\subset C(\overline{\Omega})$. Then there exists a constant $c_1 > 0$ independent of h such that, for any $u \in H^{1+s}(\Omega)$, the following estimate holds:*

$$|\nabla(u - \Pi^{\mathcal{C}} u)|_{L^2(\Omega)} \leq c_1 h^s |u|_{H^{1+s}(\Omega)}. \tag{6.2.13}$$

The proof of Theorem 6.2.1 is less technical than the one of Theorem 6.2.2, but it follows the same steps. Next we focus on the proof of Theorem 6.2.2. In view of that, we need several preliminary results.

Lemma 6.2.1. *Let $T \subset \mathbb{R}^N$ be a compact, conex set, with non-empty interior and $u \in H^{1+s}(T)$. If $F : \mathbb{R}^N \to \mathbb{R}^N$, $F(\hat{x}) = B\hat{x} + b$ with $detB \neq 0$ is an affine and invertible application and $\hat{T} = F^{-1}(T)$, we define $\hat{u} = F \circ u$. Then $\hat{u} \in H^{1+s}(\hat{T})$ and there exists a constant $C = C(N)$ such that*

$$|u|^2_{H^1(T)} \leq C||B^{-1}||^2||B||^N|u|^2_{H^1(\hat{T})} \tag{6.2.14}$$

$$|\hat{u}|^2_{H^{1+s}(\hat{T})} \leq C||B^{-1}||^{2N}||B||^{2+N+2s}|u|^2_{H^{1+s}(T)} \tag{6.2.15}$$

Corolary 6.2.2. *We consider an element T of a regular triangulation $\mathcal{T}_h$ and assume that B, $\hat{T}$ are the matrix, respectively the reference polyhedron defined in Lemma 6.2.1. Combining the condition (6.2.9) and Lemma 6.2.3 we obtain*

$$||B|| = O(h), \quad ||B^{-1}|| = O(h^{-1}). \tag{6.2.16}$$

Lemma 6.2.2 (An adapted Deny-Lions Lemma)**.** *Let $\hat{T}$ be a compact conex with non-empty interior and with piecewise C^1 boundary. Then, there exists a constant $C = C(N, \hat{T})$ such that*

$$\inf_{p \in P1} ||\hat{v} + p||_{H^{1+s}(\hat{T})} \leq C|\hat{v}|_{H^{1+s}(\hat{T})}, \quad \forall \hat{v} \in H^{1+s}(\hat{T}). \tag{6.2.17}$$

Corolary 6.2.3. *Let $\hat{T}$ be a compact conex set with non-empty interior and with piecewise C^1 boundary. If $\Pi^{\mathcal{C}}$ is the Clément interpolation operator, then*

$$||\nabla(I - \Pi^{\mathcal{C}})\hat{v}||_{L^2(\hat{T})} \leq C|\hat{v}|_{H^{1+s}(\hat{T})}. \tag{6.2.18}$$

Proof of Corollary 6.2.3. $I - \Pi^{\mathcal{C}}$ is a linear and bounded operator. Moreover,

$$(I - \Pi^{\mathcal{C}})p = 0, \quad \forall p \in P1.$$

By these reasons, we have

$$||\nabla(I - \Pi^{\mathcal{C}})\hat{v}||_{L^2(\hat{T})} = ||\nabla(I - \Pi^{\mathcal{C}})(\hat{v} + \hat{p})||_{L^2(\hat{T})}, \quad \forall p \in P1.$$

Therefore, by Lemma 6.2.2, we have

$$\begin{aligned} ||\nabla(I - \Pi^{\mathcal{C}})\hat{v}||_{L^2(\hat{T})} &\leq ||I - \Pi^{\mathcal{C}}||_{\mathcal{L}(H^{1+s}, \dot{H}^1)} \inf_{\hat{p} \in P1} ||\hat{v} + \hat{p}||_{H^{1+s}(\hat{T})} \\ &\leq C||\hat{v}||_{H^{1+s}(\hat{T})}. \end{aligned}$$

□

Lemma 6.2.3 (see Lemma 7.2.3, [113])**.** *If B is the matrix of the application F defined in Lemma 6.2.1, then*

$$||B|| \leq \frac{h_T}{\rho_{\hat{T}}}, \quad ||B^{-1}|| \leq \frac{h_{\hat{T}}}{\rho_T}. \tag{6.2.19}$$

Proofs of useful lemmas

Proof of Lemma 6.2.1. Let us only prove formula (6.2.15) since it is more technical. Formula (6.2.14) can be deduced in the same way.

By the equivalence of the norms in a finite dimensional space we consider the norm

$$||B|| = \max_{1\le i\le j\le N} |B_{ij}|.$$

We apply the definition of Sobolev spaces of fractional order (e.g. [1]) and changes of variables to obtain

$$\begin{aligned}
|\hat{u}|^2_{H^{1+s}(\hat{T})} &= \sum_{|\lambda|=1} \int_{\hat{T}}\int_{\hat{T}} \frac{|D^\alpha \hat{u}(\hat{x}) - D^\alpha \hat{u}(\hat{y})|^2}{|\hat{x}-\hat{y}|^{N+2s}} d\hat{x}d\hat{y} \\
&\le C||B||^2 (\det B)^{-2} \sum_{|\lambda|=1} \int_{\hat{T}}\int_{\hat{T}} \frac{|D^\alpha u(F(\hat{x})) - D^\alpha u(F(\hat{y}))|^2}{|\hat{x}-\hat{y}|^{N+2s}} d\hat{x}d\hat{y} \\
&= C||B||^2 |\det(B^{-1})|^2 \sum_{|\lambda|=1} \int_{T}\int_{T} \frac{|D^\alpha u(x) - D^\alpha u(y))|^2}{|B^{-1}(x-y)|^{N+2s}} dxdy. \qquad (6.2.20)
\end{aligned}$$

We notice that

$$|\det(B^{-1})| \le N!||B^{-1}||^N \text{ and } |B^{-1}(x-y)| \ge \frac{1}{||B||}|x-y|.$$

Using this last inequality, we can obtain that

$$\begin{aligned}
C||B||^2 |\det(B^{-1})|^2 & \sum_{|\lambda|=1} \int_{T}\int_{T} \frac{|D^\alpha u(x) - D^\alpha u(y))|^2}{|B^{-1}(x-y)|^{N+2s}} dxdy \\
&\le C||B^{-1}||^{2N}||B||^2 \int_{T}\int_{T} \frac{|D^\alpha u(x) - D^\alpha u(y))|^2}{|(x-y)|^{N+2s}} ||B||^{N+2s} dxdy. \\
&= C||B^{-1}||^{2N}||B||^{2+N+2s}|u|^2_{H^{1+s}(T)}
\end{aligned}$$

and according to (6.2.20) the proof is complete. □

Proof of Lemma 6.2.2. Firstly let us prove the following preliminary result.

Proposition 6.2.2. *The norm in $||\cdot||_{H^{1+s}(\Omega)}$ is equivalent with the semi-norm*

$$||\hat{v}||_\star = \Big[|\hat{v}|^2_{H^{1+s}(\hat{T})} + \sum_{|\alpha|\le 1}\Big(\int_{\hat{T}} D^\alpha \hat{v} d\hat{x}\Big)^2\Big]^{1/2}$$

□

Proof of Proposition 6.2.2. It suffices to prove that

$$||\hat{v}||_{H^1(\hat{T})} \le C\Big[|\hat{v}|^2_{H^{1+s}(\hat{T})} + \sum_{|\alpha|\le 1}\Big(\int_{\hat{T}} D^\alpha \hat{v} d\hat{x}\Big)^2\Big]$$

We argue by contradiction. Supposing that this is not true, there exists a sequence $(\hat{v}_n)_n \in H^{1+s}(\hat{T})$ such that

(I). $||\hat{v}_n||_{H^{1+s}(\hat{T})} \to 0$;

(II). $\int_{\hat{T}} D^\alpha \hat{v}_n d\hat{x} \to 0, \quad \forall |\alpha| \leq 1$;

(III). $||\hat{v}_n||_{H^1(\hat{T})} = 1$;

as n goes to infinity. From (I)-(III) the norms $||\hat{v}_n||_{H^{1+s}(\hat{T})}$ are uniformly bounded. Therefore, up to a subsequence, there exists $v \in H^{1+s}(\hat{T})$ such that

$$\hat{v}_n \rightharpoonup \hat{v} \text{ weakly in } H^{1+s}(\hat{T}).$$

By the compactness of the embedding $H^{1+s}(\hat{T}) \subset H^1(\hat{T})$ we obtain

$$\hat{v}_n \to \hat{v} \text{ strongly in } H^1(\hat{T}). \tag{6.2.21}$$

The convergence (6.2.21) and (III) imply that

$$||\hat{v}||_{H^1(\hat{T})} = 1. \tag{6.2.22}$$

On the other hand, by the inferior semi-continuity of the seminorm $|\cdot|_{H^{1+s}(\hat{T})}$ we have

$$|\hat{v}|_{H^{1+s}(\hat{T})} \leq \liminf_{n\to\infty} |\hat{v}_n|_{H^{1+s}(\hat{T})},$$

and therefore $|\hat{v}|_{H^{1+s}(\hat{T})} = 0$. This gets to $D\hat{v} = C$, where C is a constant. On the other hand, passing to the limit in (II) we obtain $\int_{\hat{T}} D^\alpha \hat{v} dx = 0$, for all $|\alpha| \leq 1$. This implies $v \equiv 0$ which is in contradiction with (6.2.22). $\square$

Now, given $\hat{v} \in H^{1+s}(\hat{T})$, let us consider $\hat{p}$ to be the unique first-order polynomial such that

$$\int_{\hat{T}} D^\alpha \hat{v} d\hat{x} = -\int_{\hat{T}} D^\alpha \hat{p} d\hat{x}, \quad \forall |\alpha| \leq 1. \tag{6.2.23}$$

Then by Proposition 6.2.2 and the identity (6.2.23) we can write

$$\inf_{p \in P1} ||\hat{v} + p||_{H^{1+s}(\hat{T})} \leq ||\hat{v} + \hat{p}||_{H^{1+s}(\hat{T})} \leq C||\hat{v} + \hat{p}||_\star = C|\hat{v}|_{H^{1+s}(\hat{T})}. \tag{6.2.24}$$

The extremes of (6.2.24) complete the proof of Lemma 6.2.2.

Proof of the main result

Proof of Theorem 6.2.2. Firstly we write

$$||u - \Pi^{\mathcal{C}} u||^2_{H^1(\Omega)} = \sum_{T \in \mathcal{T}_h} ||u - \Pi^{\mathcal{C}} u||^2_{H^1(T)}.$$

Applying Lemma 6.2.1 for any $T \in \mathcal{T}_h$ (where B_T the matrix transform from the reference polyhedron $\hat{T}$ to T) we obtain

$$||u - \Pi^{\mathcal{C}} u||^2_{H^1(\Omega)} \leq C \sum_{t \in \mathcal{T}_h} ||B_T^{-1}||^2 ||B_T||^N |\hat{u} - \Pi^{\mathcal{C}} \hat{u}|_{H^1(\hat{T})}.$$

From above and Corollary 6.2.3 we get

$$||u - \Pi^{\mathcal{C}} u||^2_{H^1(\Omega)} \le C \sum_{t \in \mathcal{T}_h} ||B_T^{-1}||^2 ||B_T||^N |\hat{u}|^2_{H^{1+s}(\hat{T})}.$$

Apply Lemma 6.2.1 for the right hand side, we have

$$C \sum_{t \in \mathcal{T}_h} ||B_T^{-1}||^2 ||B_T||^N |\hat{u}|^2_{H^{1+s}(\hat{T})} \le C \sum_{t \in \mathcal{T}_h} ||B_T^{-1}||^{2+2N} ||B_T||^{2+2N+2s} |u|^2_{H^{1+s}(T)},$$

and consequently,

$$||u - \Pi^{\mathcal{C}} u||^2_{H^1(\Omega)} \le C \sum_{t \in \mathcal{T}_h} ||B_T^{-1}||^{2+2N} ||B_T||^{2+2N+2s} |u|^2_{H^{1+s}(T)}. \tag{6.2.25}$$

Now it suffices to use the equation (6.2.16) from Corollary 6.2.2 to conclude the proof of Theorem 6.2.2:

$$||u - \Pi^{\mathcal{C}} u||^2_{H^1(\Omega)} \le C h^{2s} \sum_{T \in \mathcal{T}_h} |u|^2_{H^{1+s}(T)} = C h^{2s} |u|^2_{H^{1+s}(\Omega)}.$$

□

6.3. Adaptive FEM

6.3.1. A priori estimates in one dimension

When applying standard FEM with P1 elements and uniform meshes of size h, the dimension of the stiffness matrix for the problem (6.1.1) necessary to obtain an error h is $n(h) \sim O(h^{-1/s})$.

The goal of this section could be stated as follows: taking into account that the solution u of problem (6.1.1) belongs to $H^{1+s}(0,1)$ for some $0 < s = s(\lambda) < 1$, could we reduce the number of unknowns $n(h) \sim h^{-1/s}$ in order to get a prescribed error h by means of heterogeneous meshes? In the sequel we respond to this question.

Let us consider the partition $(0,1) = \cup_{i=0}^{\overline{n}(h)} I_i$ with $I_i = [x_i, x_{i+1}]$. Due to Corollary 6.2.1 we obtain the a priori estimate

$$|u - u_{\overline{h}}|^2_{H^1(0,1)} \lesssim |I_0|^{2s} |u|^2_{H^{1+s}(I_0)} + \sum_{i=1}^{\overline{n}(h)} |I_i|^2 |u|^2_{H^2(I_i)}, \tag{6.3.1}$$

For a given $h > 0$ small enough, we aim to build the family $(I_i)_{i \in \overline{n}(h)}$ of disjoint intervals and to determine and the number of unknowns $\overline{n}(h)$ satisfing

$$\text{error} := |I_0|^{2s} |u|^2_{H^{1+s}(I_0)} + \sum_{i=1}^{\overline{n}(h)} |I_i|^2 |u|^2_{H^2(I_i)} \lesssim h^2 \tag{6.3.2}$$

and

$$\lim_{h \to 0} \frac{\overline{n}(h)}{h^{-1/s}} = 0,$$

Our main result in this section states

Theorem 6.3.1. *Let $h > 0$ be fixed and consider in* (6.1.1) $f \in L^2(0,1)$. *Assume that u is the solution of (6.1.1) and u_n is the solution of the finite approximation (6.2.3). Then, there exists $\overline{n} = \overline{n}(h)$ and a heteregenous mesh $(I_i)_{i=1,\overline{n}(h)}$ depending on h such that*

$$\limsup_{h\to 0} \frac{\overline{n}(h)}{h^{-1}} = O(1),$$

and

$$||u - u_h||_{H^1(0,1)} \lesssim h \tag{6.3.3}$$

We show the proof of Theorem 6.3.1 in the range of λ such that $0 < \sqrt{\lambda_\star - \lambda} < 1$. The case $\sqrt{\lambda_\star - \lambda} \geq 1$ requires different computations, but the final results are similar to those in the case $0 < \sqrt{\lambda_\star - \lambda} < 1$.

Preliminaries

For $i \geq 1$, applying (5.2.9) and Hölder inequality, we obtain

$$||u''||^2_{L^2(I_i)} \lesssim \int_{I_i} f^2 dr + \int_{I_i} r^{-3+2\sqrt{\lambda_\star-\lambda}} dr + \int_{I_i} r^{-1} dr. \tag{6.3.4}$$

Without losing generality we can assume

$$|f(r)| < r^{-\frac{1}{2}}, \quad \text{a.e. in } (0,1). \tag{6.3.5}$$

Therefore, from (6.3.4) and (6.3.5) we obtain

$$||u''||^2_{L^2(I_i)} \lesssim \int_{I_i} r^{-1} dr + \int_{I_i} r^{-3+2\sqrt{\lambda_\star-\lambda}} dr, \quad \forall i \geq 1. \tag{6.3.6}$$

The case $0 < \sqrt{\lambda_\star - \lambda} < 1$. From (6.3.4) we have

$$||u''||^2_{L^2(I_i)} \lesssim \int_{I_i} r^{-3+2\sqrt{\lambda_\star-\lambda}} dr, \quad \forall i \geq 1,$$

and then

$$|u|^2_{H^2(I_i)} \lesssim \int_{x_i}^{x_{i+1}} r^{-3+2\sqrt{\lambda_\star-\lambda}} dr \lesssim |I_i|(|I_0| + \ldots + |I_{i-1}|)^{-3+2\sqrt{\lambda_\star-\lambda}}. \tag{6.3.7}$$

Moreover, for the proof of Theorem 6.3.1 we need the following additional result:

Proposition 6.3.1. *It holds that*

$$|u|^2_{H^{1+s}(I_0)} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}-2s}.$$

Proof of Proposition 6.3.1. Firstly, we have

$$\begin{aligned} u' &= \frac{1}{2\sqrt{\lambda_\star - \lambda}} \Big(\frac{1}{2} + \sqrt{\lambda_\star - \lambda}\Big)(G_2(0) - G_1(r)) r^{-\frac{1}{2}+\sqrt{\lambda_\star-\lambda}} \\ &+ \frac{1}{2\sqrt{\lambda_\star - \lambda}} \Big(\frac{1}{2} - \sqrt{\lambda_\star - \lambda}\Big)(G_2(r) - G_2(0)) r^{-\frac{1}{2}-\sqrt{\lambda_\star-\lambda}}. \end{aligned} \tag{6.3.8}$$

From (6.3.8) we obtain

$$|u'(r)| \lesssim |J_1| + |J_2| + |J_3| \tag{6.3.9}$$

where

$$J_1 = r^{-1/2+\sqrt{\lambda_\star-\lambda}}, \quad J_2 = G_1(r) r^{-1/2+\sqrt{\lambda_\star-\lambda}}, \quad J_3 = (G_2(r) - G - 2(0)) r^{-1/2-\sqrt{\lambda_\star-\lambda}}.$$

Then we obtain

$$|J_1|^2_{H^s(I_0)} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}-2s}.$$

Following the proof of the regularity of J_2 in (5.2.36) we have

$$|J_2|^2_{H^s(I_0)} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}-2s}, \text{ for any } 0 < s < \sqrt{\lambda_\star - \lambda}.$$

For J_3 we have the estimate

$$\begin{aligned} J_3'(r) &= f(r) + C(G_2(r) - G_2(0)) r^{-\frac{3}{2}-\sqrt{\lambda_\star-\lambda}} \\ &\lesssim |f(r)| + r^{-\frac{1}{2}}. \end{aligned} \tag{6.3.10}$$

Therefore, $J_3 \in W^{1,p}(0,1)$, $\forall p < 2$. Due to the continuous embedding $W^{2,p}(0,1) \hookrightarrow H^{1+s}(0,1)$, with $s = 3/2 - 1/p$ we obtain $|u|_{H^{1+s}(I_0)} \lesssim ||u||_{W^{2,p}(I_0)}$. On the other hand, we have

$$\begin{aligned} ||u||^2_{W^{2,p}(I_0)} &\lesssim |u''|^2_{L^p(I_0)} + |u'|^2_{L^p(I_0)} + |u|^2_{L^p(I_0)} \\ &\lesssim \Big(\int_0^{|I_0|} (|f(r)| + r^{-1/2})^p dr \Big)^{2/p} \lesssim \Big(\int_0^{|I_0|} r^{-p/2} dr \Big)^{2/p} \\ &\lesssim |I_0|^{2/p-1} = |I_0|^{2-2s} \end{aligned} \tag{6.3.11}$$

and, because $\sqrt{\lambda_\star - \lambda} < 1$, we get

$$|u|^2_{H^{1+s}(I_0)} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}-2s}$$

which ends the proof. □

6.3.2. Proof of Theorem 6.3.1

In the following we denote

$$\text{error}_0 := |I_0|^{2s} |u|^2_{H^{1+s}(I_0)}, \quad \text{error}_i := |I_i|^2 |u|^2_{H^2(I_i)} \quad \forall i \geq 1.$$

Then from (6.3.7) and Lemma (6.3.1) we obtain

$$\text{errror}_0 \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}}, \tag{6.3.12}$$

$$\text{error}_i \lesssim |I_i|^3 (|I_0| \ldots + |I_{i-1}|)^{-3+2\sqrt{\lambda_\star-\lambda}}, \quad i \geq 1. \tag{6.3.13}$$

Next we build the intervals $(I_i)_i$. The idea concerns to impose the error to be uniform distributed on each interval i.e.,

$$\text{error}_0 \sim \text{error}_1 \sim \ldots \sim \text{error}_{\overline{N}} \sim \mathcal{C} \quad \text{(constant)}.$$

We proceed in several steps as follows.

a). Let $h > 0$ to be a fixed prescribed error.

b). We define $\mathcal{C}$ as $\mathcal{C} = h^2/(\overline{n}+1)$, where $\overline{N}$ is unknown a priori.

c). Choose I_0: we consider I_0 such that $\mathcal{C} \sim |I_0|^{2\sqrt{\lambda_\star - \lambda}}$ i.e.,

$$|I_0| = \Big(\frac{h^2}{\overline{n}+1}\Big)^{1/2\sqrt{\lambda_\star-\lambda}}. \tag{6.3.14}$$

d). Choose I_i, with $i \in \{1, 2, \ldots \overline{n}\}$ such that

$$|I_i|^3(|I_0| + \ldots + |I_i|)^{-3+2\sqrt{\lambda_\star-\lambda}} \le |I_0|^{2\sqrt{\lambda_\star-\lambda}} \le |I_i|^3(|I_0| \ldots + |I_{i-1}|)^{-3+2\sqrt{\lambda_\star-\lambda}}, \quad \forall i. \tag{6.3.15}$$

More precisely, if we impose

$$a_i|I_0| \le |I_i| \le b_i|I_0|, \quad i \ge 1, \tag{6.3.16}$$

where

$$\begin{cases} a_i = (b_0 + b_1 + \ldots + b_{i-1})^{1-2/3\sqrt{\lambda_\star-\lambda}}, \\ b_i = (a_0 + a_1 + \ldots + a_i)^{1-2/3\sqrt{\lambda_\star-\lambda}}, \end{cases} \tag{6.3.17}$$

and $a_0 = b_0 = 1$, then the relation (6.3.15) holds. Consequently, we may choose

$$|I_i| = \frac{a_i + b_i}{2}|I_0|. \tag{6.3.18}$$

e). Let us prove that the choice of the mesh depicted above ensures

$$\text{error} = |I_0|^{2s}|u|^2_{H^{1+s}(I_0)} + \sum_{i=1}^{\overline{n}} |I_i|^2|u|^2_{H^2(I_i)} \lesssim h^2, \quad \forall \quad \overline{n} \in \mathbb{N}.$$

Indeed, note that for all $i \in 1, \overline{n}$

$$(|I_0| \ldots + |I_{i-1}|)^{-3+2\sqrt{\lambda_\star-\lambda}} \le \Big(1 + \frac{b_i}{1 + a_1 + \ldots a_{i-1}}\Big)^{3-2\sqrt{\lambda_\star-\lambda}}(|I_0| + \ldots + |I_i|)^{-3+2\sqrt{\lambda_\star-\lambda}}. \tag{6.3.19}$$

Moreover, by Lemma 6.3.1 below, we obtain that

$$\frac{b_i}{1 + a_1 + \ldots a_{i-1}} \to 0, \text{ as } i \to \infty \tag{6.3.20}$$

From (6.3.7) we have

$$\text{error} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}} + \sum_{i=1}^{\overline{n}(h)} |I_0|^3(|I_0| + \ldots + |I_{i-1}|)^{-3+2\sqrt{\lambda_\star-\lambda}}.$$

From (6.3.19) and (6.3.20) we obtain

$$\text{error} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}} + \sum_{i=1}^{\overline{n}(h)} |I_0|^3(|I_0| + |I_1| + \ldots + |I_i|)^{-3+2\sqrt{\lambda_\star-\lambda}}.$$

Therefore, by (6.3.15), we conclude

$$\text{error} \lesssim |I_0|^{2\sqrt{\lambda_\star-\lambda}} + \sum_{i=1}^{\overline{n}} |I_0|^{2\sqrt{\lambda_\star-\lambda}} \lesssim h^2. \tag{6.3.21}$$

The asymptotic value of $\overline{n}(h)$: The number $\overline{n}(h)$ is chosen as the biggest natural number $\overline{n}$ such that

$$\sum_{i=0}^{\overline{n}(h)} |I_i| = |I_0| \sum_{i=0}^{\overline{n}(h)} \frac{a_i + b_i}{2} \leq \mathrm{diam}(I), \quad I = [0,1], \quad \text{as } h \to 0, \tag{6.3.22}$$

or equivalently

$$|I_0| \sum_{i=0}^{\overline{n}(h)} \frac{a_i + b_i}{2} \sim \mathrm{diam}(I), \quad \text{as } h \to 0. \tag{6.3.23}$$

In order to determine the asymptotic value of $\overline{n}(h)$, as $h \to 0$, we need in advance to prove the asymptotic behavior of the sequences a_n, b_n defined in (6.3.17). To simplify the computations, we make the notation $s := \sqrt{\lambda_\star - \lambda}$, and $\tilde{s} =: 1 - 2s/3$ and we take into account that $s \in (0,1)$ implies $\tilde{s} \in (1/3, 1)$. We claim

Lemma 6.3.1. *There exist four constants $C_1(s), C_2(s), D_1(s), D_2(s)$ independent of n such that*

$$C_1(s) n^{\frac{3-2s}{2s}} \leq a_n \leq C_2(s) n^{\frac{3-2s}{2s}}, \quad \forall n \in \mathbb{N}, \tag{6.3.24}$$

respectively

$$D_1(s) n^{\frac{3-2s}{2s}} \leq b_n \leq D_2(s) n^{\frac{3-2s}{2s}}, \quad \forall n \in \mathbb{N}, \tag{6.3.25}$$

From (6.3.14), (6.3.23), Lemma 6.3.1 and (6.3.30) (in the proof of Lemma 6.3.1) we obtain

$$\begin{aligned} 1 = &\mathrm{diam}(I) \sim |I_0| \sum_{i=1}^{\overline{n}(h)} \frac{a_i + b_i}{2} \sim \Big(\frac{h^2}{\overline{n}(h)}\Big)^{\frac{1}{2s}} \sum_{i=1}^{\overline{n}(h)} i^{\frac{3-2s}{2s}} \\ &\sim \Big(\frac{h^2}{\overline{n}(h)}\Big)^{\frac{1}{2s}} \overline{n}(h)^{\frac{3-2s}{2s}+1} \sim h^{\frac{1}{s}} \overline{n}(h)^{\frac{1}{s}} \end{aligned} \tag{6.3.26}$$

From (6.3.26) we obtain

$$\overline{n}(h) \sim h^{-1}. \tag{6.3.27}$$

Proof of Lemma 6.3.1. It suffices to prove the validity of (6.3.24). The behavior of b_n yields immediately as a consequence. Let us proof the easiest part of (6.3.24), i.e., the upper bound of a_n. We proceed by induction. In fact, we are going to prove the estimate below for all $n \in \mathbb{N}$:

$$a_n \leq C_2(s)(n+1)^{\frac{\tilde{s}}{1-\tilde{s}}}, \tag{6.3.28}$$

where the constant $C_2(s)$ is big enough, but its election will be specified later.

For $n = 0, 1$, the inequality (6.3.28) comes for free, once we choose $C_2(s) \geq 1$. Let us now assume that (6.3.28) is true for all natural numbers $k \leq n$, with $n \geq 2$, and we are looking for the constant $C_2(s)$ sufficiently large, such that (6.3.28) should be true also for $k = n+1$. Using the expression of b_n, the sequence a_n could be rewritten as

$$\begin{aligned} a_{n+1} &= (a_0^{\tilde{s}} + (a_0 + a_1)^{\tilde{s}} + \ldots + (a_0 + a_1 + \ldots a_n)^{\tilde{s}})^{\tilde{s}} \\ &:= (S_0^{\tilde{s}} + S_1^{\tilde{s}} + \ldots S_n^{\tilde{s}})^{\tilde{s}}. \end{aligned} \tag{6.3.29}$$

Using the limit
$$\lim_{n\to\infty}\frac{1^\alpha+2^\alpha+\ldots+n^\alpha}{n^{\alpha+1}}=\frac{1}{1+\alpha},\quad \forall\alpha>0,$$
there exist positive constants $L_1(\alpha), L_2(\alpha)$ such that
$$L_1(\alpha)n^{1+\alpha}\leq 1^\alpha+2^\alpha+\ldots+n^\alpha\leq L_2(\alpha)n^{1+\alpha},\quad \forall n\in\mathbb{N},\alpha>0. \tag{6.3.30}$$
Let us call $L_1(\alpha), L_2(\alpha)$ to be the optimal constants in the above inequality. From this and the induction hypothesis we obtain
$$S_k\leq C_2(s)L_2\Big(\frac{\tilde{s}}{1-\tilde{s}}\Big)k^{\frac{1}{1-\tilde{s}}},\quad \forall k\leq n.$$
Combining this with (6.3.29), we have
$$a_{n+1}\leq C_2^{\tilde{s}^2}(s)L_2^{\tilde{s}^2}\Big(\frac{\tilde{s}}{1-\tilde{s}}\Big)\Big(1^{\frac{\tilde{s}}{1-\tilde{s}}}+\ldots+n^{\frac{\tilde{s}}{1-\tilde{s}}}\Big)^{\tilde{s}}$$
Applying again (6.3.30), we find out
$$a_{n+1}\leq C_2^{\tilde{s}^2}(s)L_2^{\tilde{s}(\tilde{s}+1)}\Big(\frac{\tilde{s}}{1-\tilde{s}}\Big)n^{\frac{\tilde{s}}{1-\tilde{s}}}.$$
Choosing $C_2(s)\geq\max\Big\{1,L_2^{\frac{\tilde{s}}{1-\tilde{s}}}\big(\frac{\tilde{s}}{1-\tilde{s}}\big)\Big\}$, we obtain
$$a_{n+1}\leq C_2(s)n^{\frac{\tilde{s}}{1-\tilde{s}}}$$
which completes the proof of the upper bound.

Let us now analyze the lower bound of the sequence a_n. Let us assume that
$$a_n>C(\alpha)n^\alpha,\qquad \text{for any } n\in\mathbb{N}. \tag{6.3.31}$$
On the other hand, we can obtain particularly the estimate
$$1+2^\alpha+\ldots+(n-1)^\alpha\geq\frac{1}{4}\Big(\frac{n}{2}\Big)^{\alpha+1},\quad \forall n. \tag{6.3.32}$$
Then
$$\begin{aligned}a_n&\geq C^{\tilde{s}^2}(\alpha)(1+\ldots+(1+\ldots+(n-1)^\alpha)^{\tilde{s}})^{\tilde{s}}\\&\geq C^{\tilde{s}^2}(\alpha)\Big\{\Big[\frac{1}{4}\Big(\frac{1}{2}\Big)^{\alpha+1}\Big]^{\tilde{s}}+\ldots+\Big[\frac{1}{4}\Big(\frac{n}{2}\Big)^{\alpha+1}\Big]^{\tilde{s}}\Big\}^{\tilde{s}}\\&=C^{\tilde{s}^2}(\alpha)\Big(\frac{1}{2}\Big)^{(\alpha+3)\tilde{s}^2}\Big[1+\ldots+n^{(\alpha+1)\tilde{s}}\Big]^{\tilde{s}}.\end{aligned}$$
Applying once again inequality (6.3.32), we obtain
$$a_n\geq C^{\tilde{s}^2}(\alpha)\Big(\frac{1}{2}\Big)^{(\alpha+3)\tilde{s}^2+((\alpha+1)\tilde{s}+3)\tilde{s}}n^{((\alpha+1)\tilde{s}+1)\tilde{s}},\quad \forall n\in\mathbb{N}. \tag{6.3.33}$$

The first conclusion says that, whenever (6.3.31) holds, then (6.3.33) also holds. For instance, it can be observed that (6.3.31) holds with $(\alpha, C(\alpha)) = (\tilde{s}, 1)$. Motivated by this, we may introduce by iteration a sequence $(\alpha_p, C_p(\alpha_p))$, $p \geq 1$, which is defined as

$$\begin{cases} \alpha_1 = \tilde{s}, \quad C_1(\alpha_1) = 1; \\ \alpha_{p+1} = ((\alpha_p + 1)\tilde{s} + 1)\tilde{s} \\ C_{p+1}(\alpha_{p+1}) = C_p^{\tilde{s}^2}(\alpha_p)\Big(\frac{1}{2}\Big)^{(\alpha_p+3)\tilde{s}^2+((\alpha_p+1)\tilde{s}+3)\tilde{s}}, \quad \forall p \geq 1, \end{cases} \tag{6.3.34}$$

satisfying

$$a_n \geq C_p(\alpha_p) n^{\alpha_p}, \quad \forall n, \forall p. \tag{6.3.35}$$

By the recursive definition of $(\alpha_p)_p$ we deduce its general form

$$\alpha_p = \tilde{s} + \ldots + \tilde{s}^{2p-1} = \frac{\tilde{s} - \tilde{s}^{2p}}{1 - \tilde{s}}.$$

Then

$$\alpha_p \to \frac{\tilde{s}}{1 - \tilde{s}}, \text{ as } p \to \infty, \tag{6.3.36}$$

thus $\alpha_p \leq L(s)$, $\forall p$. Then, from (6.3.35), we have

$$\begin{aligned} C_{p+1}(\alpha_{p+1}) &\geq C_p^{\tilde{s}^2}(\alpha_p)\Big(\frac{1}{2}\Big)^{(L(s)+3)\tilde{s}^2+((L(s)+1)\tilde{s}+3)\tilde{s}} \\ &:= D(s) C_p^{\tilde{s}^2}(\alpha_p) \quad \forall p \geq 1. \end{aligned}$$

From above we may obtain

$$C_p(\alpha_p) \geq D(s)^{\frac{1-\tilde{s}^{2p}}{1-\tilde{s}^2}}.$$

Therefore

$$C_1(s) := \liminf_{p \to \infty} C_p(\alpha_p) > 0.$$

By this and (6.3.36), passing to liminf as $p \to \infty$ in (6.3.35), we obtain

$$a_n \geq C_1(s) n^{\frac{\tilde{s}}{1-\tilde{s}}} = C_1(s) n^{\frac{3-2s}{2s}}, \quad \forall n \in \mathbb{N},$$

and the conclusion follows. Theorem 6.3.1 is proved. □

6.3.3. Numerical experiments

As remarked in Figure 6.4, since the value $\lambda = 0.2$ is getting close to the critical Hardy constant $\lambda_\star = 0.25$, the standard FEM requires much bigger computational cost than in the previous case $\lambda = -0.3$. Therefore the number of unknowns $\overline{n}(h)$ is much larger than those corresponding to the AFEM (in red) respectively the FEM approximation of the standard Laplacian (in green). For that reason, the blue plot does not appear in the picture of the left hand side.

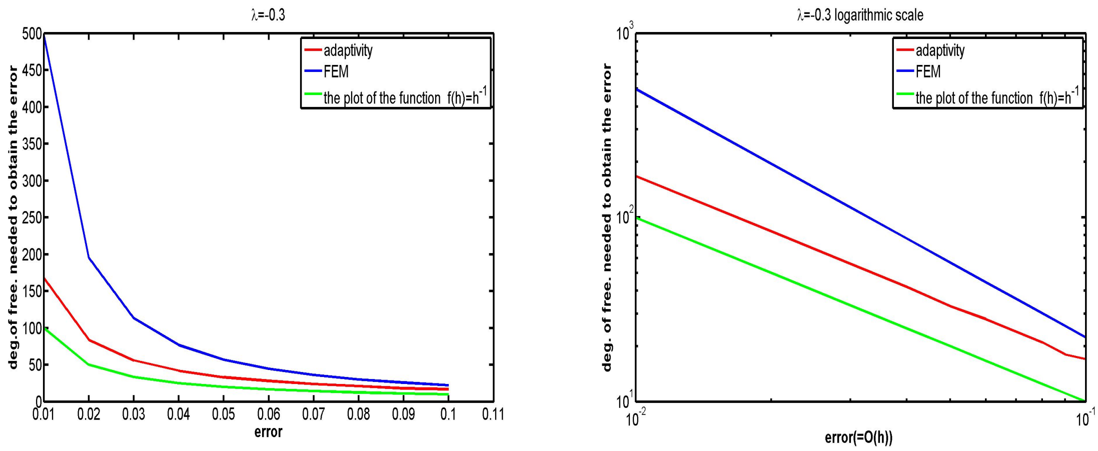

Figure 6.2: (On the left) The number of unknowns $\overline{n}(h)$ needed to reach a prescribed error h comparing three approximations: standard FEM for $\lambda = -0.3$ (in blue), standard FEM for $\lambda = 0$ (in green), adapted FEM for $\lambda = -0.3$ (in red). (On the right) The number of unknowns $\overline{n}(h)$ in logarithmic scale for $\lambda = -0.3$.

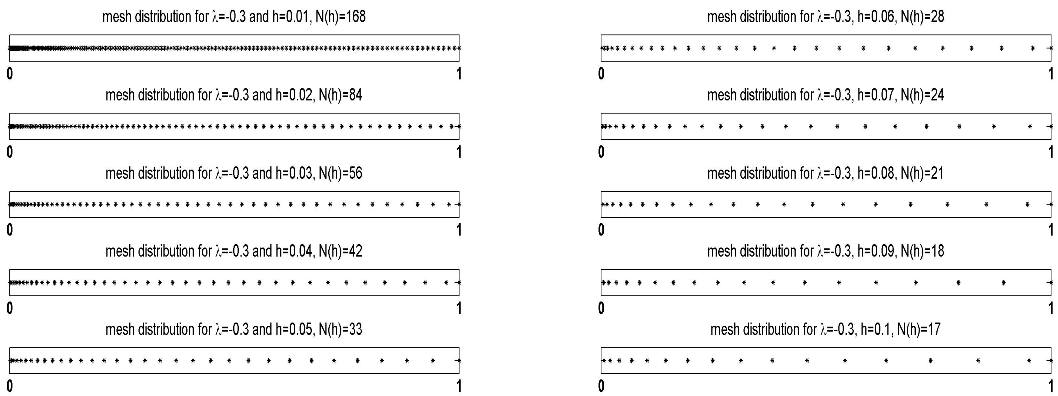

Figure 6.3: The mesh distribution according to the election of I_i in (6.3.17)-(6.3.18)

6.3.4. A posteriori error analysis: general aspects

Firstly, let us briefly remind the results obtained in the previous section when applying the standard FEM. Assume u is a variational solution of (6.1.1) and u_h is its finite element approximation corresponding to a regular mesh $\mathcal{T}_h$. We have shown that

$$||u - u_h||_{H_0^1(\Omega)} \leq c_1 h^s ||u||_{H^{1+s}(\Omega)}, \tag{6.3.37}$$

for some constant $0 < s < 1$, depeding on λ. Therefore, in order to reach a prescribed error of order h it is necessary to solve $(1/h)^{N/s}$ number of unknowns. Since

$$\left(\frac{1}{h}\right)^{\frac{N}{s}} >> \left(\frac{1}{h}\right)^{N}, \quad \text{provided } s << 1,$$

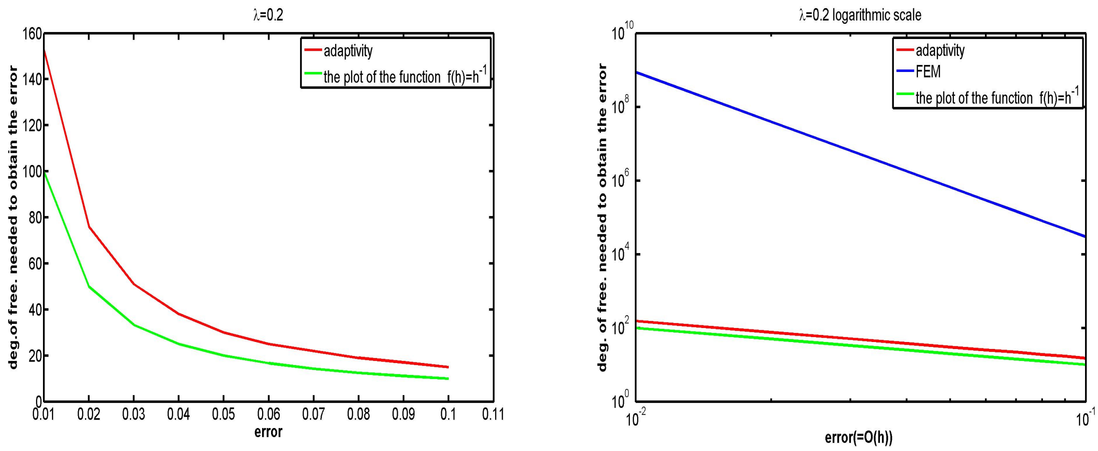

Figure 6.4: (On the left) The number of unknowns $\overline{n}(h)$ needed to reach a prescribed error h comparing three approximations: standard FEM for $\lambda = 0.2$ (in blue), standard FEM for $\lambda = 0$ (in green), adapted FEM for $\lambda = 0.2$ (in red); (On the right) The number of unknowns $\overline{n}(h)$ in logarithmic scale for $\lambda = 0.2$.

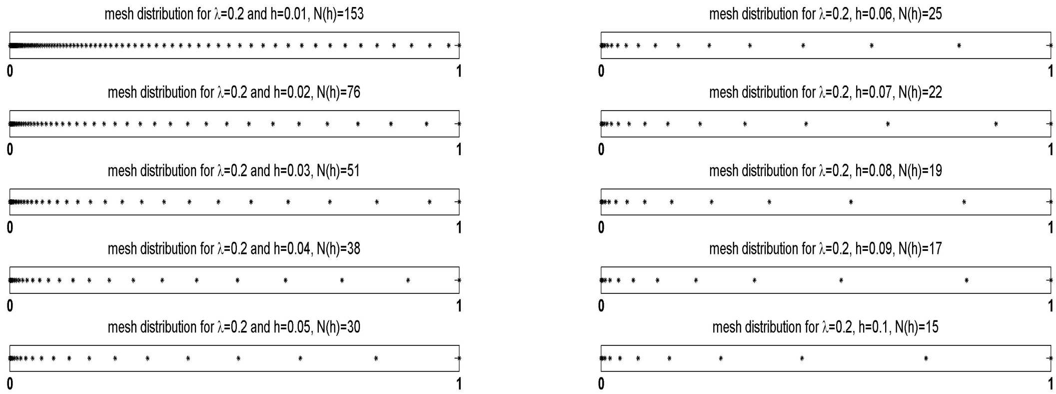

Figure 6.5: The mesh distribution according to the election of I_i in (6.3.17)-(6.3.18) for $\lambda = 0.2$

the standard FEM applied to A_λ provides slower rates of convergence than the standard FEM does for elliptic operators with standard H^2 regularity.

The goal of this section is to readapt the classical FEM in order to reduce the computational cost induced by the lack of regularity of the solution of our problem (This is expressed by the fact that $s < 1$).

AFEM is, by now, a finite element method widely used in numerical analysis of PDEs to achieve better accuracy with minimum number of degrees of freedom (unknowns). A posteriori estimations are the ingredients of adaptivity. They are computable quantities depending on the numerical solution and data, providing information about the quality of approximation and used to make mesh modifications. Our goal is to build a sequence of meshes which equi-distributes the approximation errors and implicitly optimizes the computational cost. Using error indicators, one can provide local mesh modifications leading to loaps of the form

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE/COARSEING}$$

Starting with the paper by Dörfler [49] for the Poisson equation, the error reduction was obtained in the energy norm with a preassigned tolerance in finite number of steps provided the initial mesh is sufficiently refined to resolve data and the sum of the local error indicators of elements are marked for refinement to a fixed proportion of the global error estimator.

In the sequel, we consider a PDE written in variational formulation as

$$\begin{cases} a(u,v) = (f,v), \quad \forall v \in V, \\ u \in V. \end{cases} \tag{6.3.38}$$

Here V denotes a Hilbert space and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bilinear form satisfying the conditions required by the Lax-Migram Lemma (coercivity, continuity, symmetry). In particular, for problem (6.1.1) we have $V = H_\lambda$ and

$$a(u,v) = \int_\Omega \Big[\nabla u \cdot \nabla v - \lambda \frac{uv}{|x|^2}\Big] dx.$$

Next let us briefly introduce the main ingredients used for aposteriori error analysis, which could be found in details in [92]. At the end of the section we will present some applications (numerical simulations) to the problem (6.1.1).

A posteriori estimates

Let $\mathcal{T}_{\overline{h}}$ be an initial configuration and such that $u_{\overline{h}}$ solve the corresponding the numerical finite element approximation of (6.3.38)

$$\begin{cases} a(u_{\overline{h}}, v_{\overline{h}}) = (f, v_{\overline{h}}), \quad \forall v_{\overline{h}} \in V_{\overline{h}}, \\ u_{\overline{h}} \in V_{\overline{h}}. \end{cases} \tag{6.3.39}$$

In addition, let $S_{\overline{h}}$ denoting the interior faces or edges of the mesh $\mathcal{T}_{\overline{h}}$.

1. **The residual.** A posteriori estimates are strongly connected to the so-called residual operator. More precisely, the residual is a linear and bounded operator, i.e. $R = R(u_{\overline{h}}) \in V^\star$ defined by
$$< R(u_{\overline{h}}), v >_{V^\star,V} = (f,v) - a(u_{\overline{h}}, v) = a(e_{\overline{h}}, v) \tag{6.3.40}$$
where $e_{\overline{h}} = u - u_{\overline{h}}$. From the variational formulations of both numerical and continuous problems, the residual satisfies
$$< R(u_{\overline{h}}), v_{\overline{h}} >_{V^\star,V} = 0, \quad \forall v_{\overline{h}} \in V_{\overline{h}}. \tag{6.3.41}$$

Then the following a priori estimates hold

$$||u - u_{\overline{h}}||_V \leq \frac{c_2}{c_1} \inf_{v \in V_{\overline{h}}} ||u - v||_V, \tag{6.3.42}$$

where c_1, c_2 are the coercivity respectively continuous constants for the bilinear form $a(\cdot, \cdot)$. On the other hand, we have the posteriori estimates for the residual:

$$c_1||u - u_{\overline{h}}||_V \leq ||R(u_{\overline{h}})||_{V^\star} \leq c_2||u - u_{\overline{h}}||_V. \tag{6.3.43}$$

Therefore, computing the error in the energy norm reduces to give estimates for the norm of the residual $||R(u_H)||_{V^\star}$. By integrations by parts, one can obtain the error representation formula

$$< R(u_{\overline{h}}), v >= \sum_{T \in \mathcal{T}_{\overline{h}}} \int_T R_T(u_{\overline{h}}) v dx + \sum_{S \in S_{\overline{h}}} \int_S J_S(u_{\overline{h}}) v d\sigma, \tag{6.3.44}$$

In our case,

$$R_T(u_{\overline{h}}) = f + \Delta u_{\overline{h}} + \frac{\lambda}{|x|^2} u_{\overline{h}}, \text{ in any } T \in \mathcal{T}_{\overline{h}},$$

is called the element residual, whereas the quantity

$$J_S(u_{\overline{h}}) = [[\nabla u_{\overline{h}}]] \cdot \nu_S, \text{ on } S \in S_{\overline{h}},$$

denotes the jump residual.

2. **Local error indicator.** In the sequel, let us define the quantities

$$\eta_{\overline{h}}(T)^2 := H_T^2 ||R_T(u_{\overline{h}})||^2_{L^2(T)} + \sum_{S \in S_{\overline{h}}} H_S ||J_S(u_{\overline{h}})||^2_{L^2(S)}.$$

Given a subset ω, we define the error estimator $\eta_{\overline{h}}(\omega)$ by

$$\eta_{\overline{h}}(\omega)^2 := \sum_{T \in \mathcal{T}, T \subset \omega} \eta_{\overline{h}}(T)^2. \tag{6.3.45}$$

Upper bounds: There exists a constant C_1 depending on $\gamma^\star, c_1, c_2$ such that

$$||u - u_{\overline{h}}||_V^2 \leq C_1 \eta_{\overline{h}}^2(\Omega). \tag{6.3.46}$$

Lower bounds: Let $\overline{R_T}$ be the L^2-projection on the space of R_T on $P_{k-1}(T)$, $k \geq 1$. Next it is necessary to introduce the oscillator by

$$osc_{\overline{h}}(T)^2 := \overline{h}_T^2 ||R_T - \overline{R_T}||^2_{L^2(T)}.$$

For a given $\omega \subset \Omega$ we denote

$$osc_{\overline{h}}(\omega)^2 := \sum_{T \in \mathcal{T}_{\overline{h}}, T \subset \omega} osc_{\overline{h}}(T)^2 \tag{6.3.47}$$

Then, there exists a constant $C_2 = C_2(\gamma^\star, C_1, C_2)$ s.t.

$$C_2 \eta_{\overline{h}}^2(T) \leq ||u - u_{\overline{h}}||_{H^1(\omega_T)} + osc_{\overline{h}}(\omega_T)^2,$$

where ω_T are all elements sharing at least a side with T. The posteriori analysis applies two marking strategies by using the bisection algorithm.

3. **Marking strategies.**

Marking strategy E: Given a parameter $0 < \theta < 1$ we construct a minimal set $\overline{\mathcal{T}}_{\overline{h}}$ of $\mathcal{T}_{\overline{h}}$ such that

$$\sum_{T\in\overline{\mathcal{T}}_{\overline{h}}} \eta_{\overline{h}}(T)^2 \geq \theta_E \eta_E(\Omega)^2 \tag{6.3.48}$$

and mark all elements of $\overline{\mathcal{T}}_{\overline{h}}$ for refinement.

Marking strategy 0: Let us consider $0 < \theta_0 < 1$ and $\overline{\mathcal{T}}_{\overline{h}}$ produced before. Then we enlarge $\overline{\mathcal{T}}_{\overline{h}}$ to a minimal set such that

$$\sum_{T\in\overline{\mathcal{T}}_{\overline{h}}} osc_{\overline{h}}(T)^2 \geq \theta_0 osc_{\overline{h}}(\Omega)^2,$$

and mark all elements in $\overline{\mathcal{T}}_H$ for refinement. Under these hypothesis, the data oscillation reduction holds and the algorithm converges.

4. **Steps in the implementation of the numerical algorithm.**
The algorithm could be briefly explained as follows.

a) Let us consider an initial mesh $\mathcal{T}_{\overline{h}}$. Then we fix $0 < tol$ and $0 < \theta_r, \theta_c < 1$. Here *tol* denotes the prescribed error to be obtained, θ_r is the refinement parameter and θ_c is the coarsening parameter

b) Compute $u_{\overline{h}}$.

c) Compute local error estimators η_T, $T \in \mathcal{T}_{\overline{h}}$ and put

$$\eta^2 := \sum_{T\in\mathcal{T}_H} \eta_T^2.$$

d) If $\eta < tol$, we STOP, otherwise we CONTINUE (marking procedure).

e) Markig procedure: Find minimal subsets $\mathcal{M}_r$, $\mathcal{M}_c$, such that

$$\sum_{T\in\mathcal{M}_r} \eta_T^2 > \theta_r \eta^2, \quad \sum_{T\in\mathcal{M}_c} \eta_T^2 < \theta_c \eta^2,$$

and η_T small enough for $T \in \mathcal{M}_c$.

f) REFINE/COARSEN: Refine triangles $T \in \mathcal{M}_r$ and coarsen triangles $T \in \mathcal{M}_c$ to generate a new mesh $\mathcal{T}$. Roughly speaking, refine where the error is "too big" and coarse where the error is "to small".

g) We enter into a new loop till we reach *tol*.

6.3.5. Numerical experiments in two dimensions and interpretation

Next we design some simulations based on the tutorial code by [42] where the above adaptive algorithm was implemented. More precisely, in the following we draw some numerical results when varying the parameter λ for the singular problem (6.1.1). We consider the two-dimensional case when $\Omega = [-1,1]\times[0,1]$ and the singularity $x = 0$ is located on the boundary of Ω. Due to the optimal Hardy inequalities discussed in Chapter 2, the problem is well-posed in the range of parameters $\lambda \leq 1$. Moreover, when λ is getting closer to 1, the solution (6.1.1) loses regularity. This fact is illustrated in the next simulations where, by means of an aposteriori analysis, heterogenous meshes appear in the neighborhood of the origin. For $\lambda > 1$ the problem (6.1.1) is ill-posed, fact emphasized in the last simulation below.

1. The case $\lambda = -0.5$: the problem is well posed in H^1 and the solution has H^2 regularity at origin. For that reason, the refinement is regular around the singularity $x = 0$ (see Figure (6.6)).
2. The case $\lambda = 0$. This case corresponds to the classical Laplacian in in which standard H^2 applies and regular meshes are used for the optimal complexity of FEM (see Figure (6.7)).
3. The case $\lambda = 0.5$. The operator A_λ becomes less coercive (but still coercive in $H_0^1(\Omega)$) and the solution to (6.1.1) is out of H^2 regularity at the origin $x = 0$. Finer refinements are needed close to $x = 0$ to approximate efficiently the numerical solution (see Figure (6.8)).
4. The case $\lambda = 1$. This corresponds to the critical Hardy constant, for which the problem (6.1.1) is still well-posed in the Hardy space H_1 (not in $H_0^1(\Omega)$). The refinements around the origin are even finer than in the previous case $\lambda = 0.5$ since the solution becomes even more singular (See Figure (6.9)).
5. The case $\lambda = 1.1$. The problem (6.1.1) is ill-posed. This can be observed in Figure (6.11), where, after 29 iterations oscillatory solutions appear. This phenomenon cannot occur in our functional setting due to the validity of the Maximum Principle.

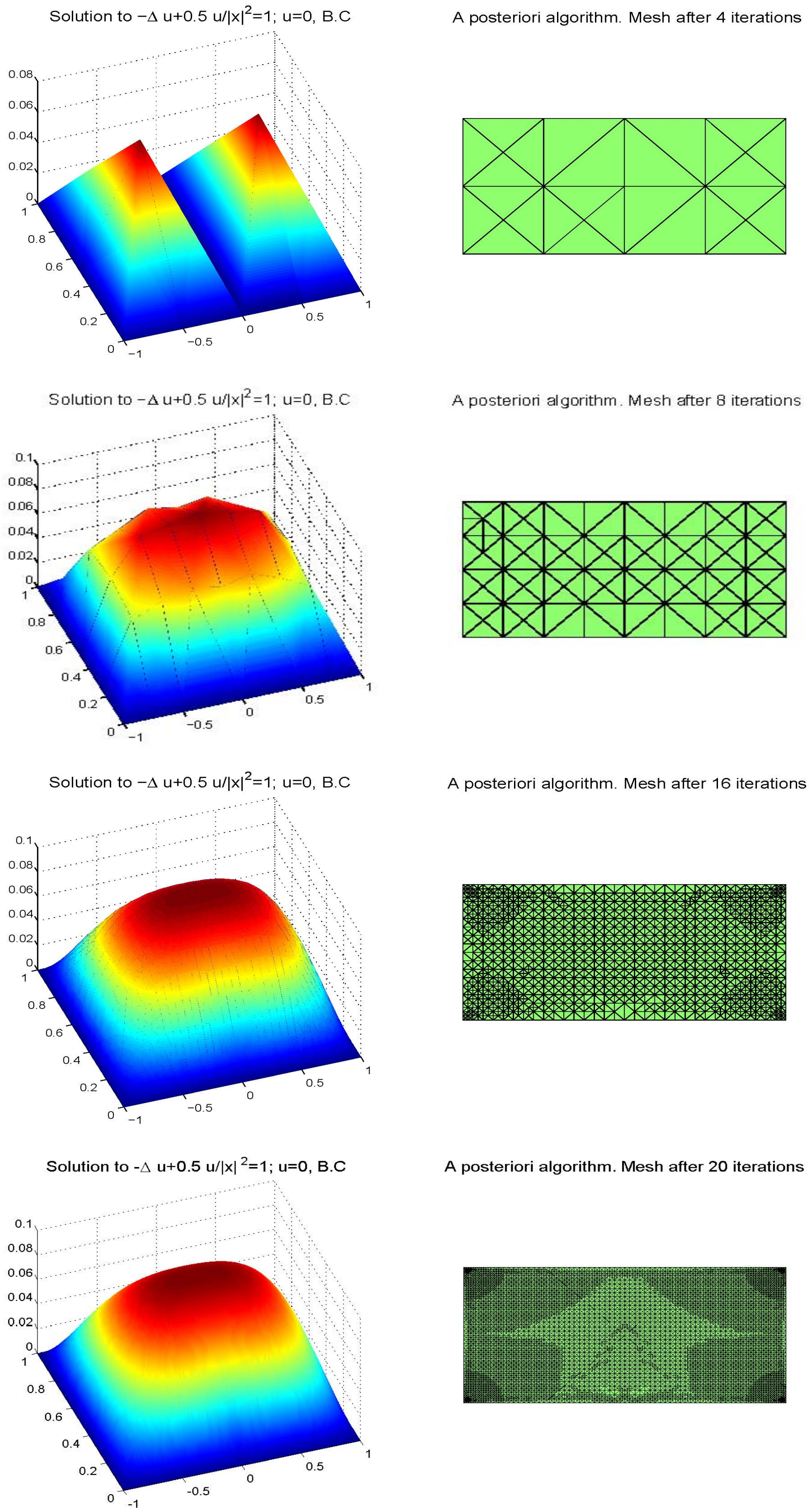

Figure 6.6: Subcritical case $\lambda = -0.5$

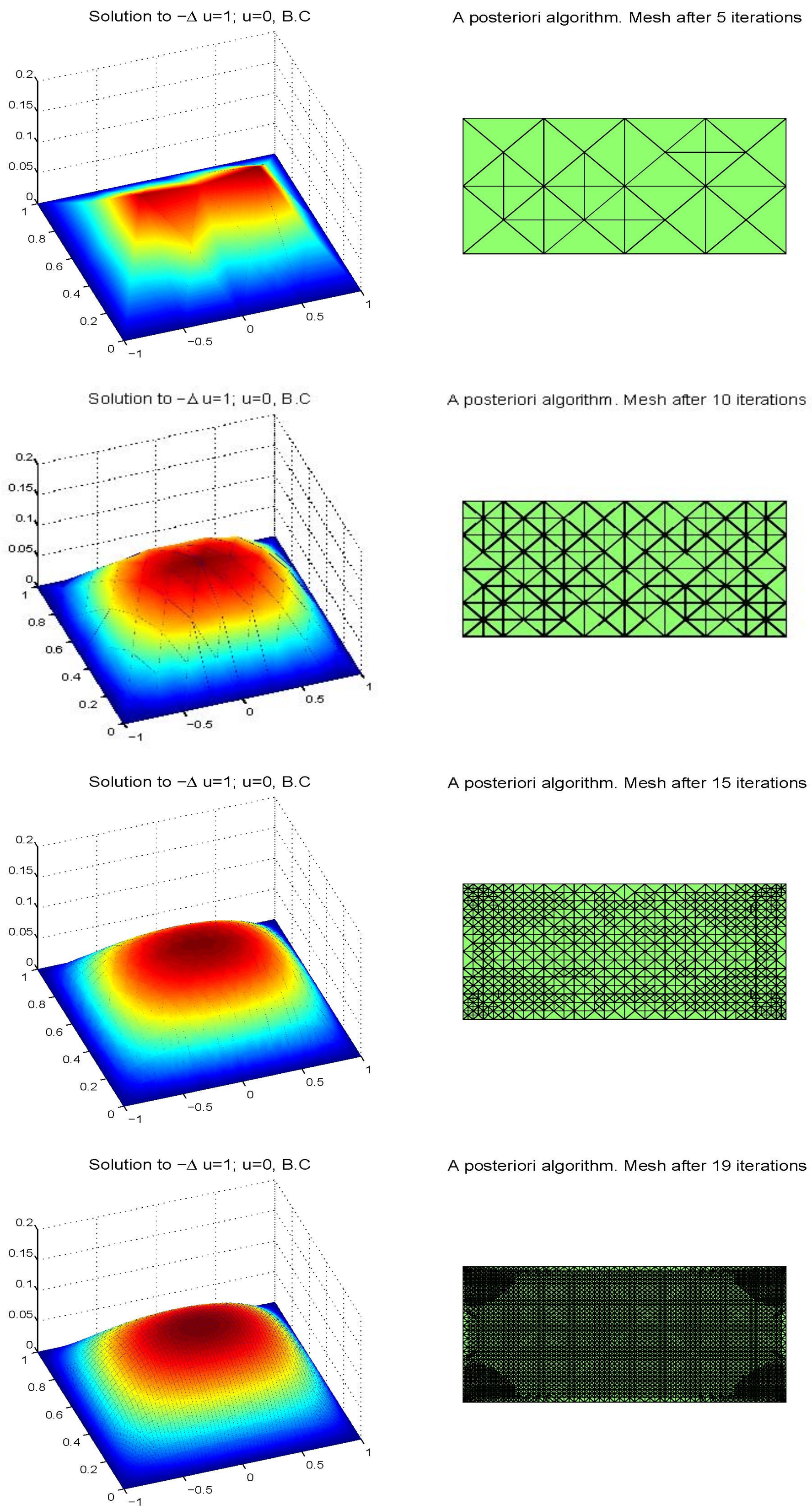

Figure 6.7: The case $\lambda = 0$ (classical Laplacian)

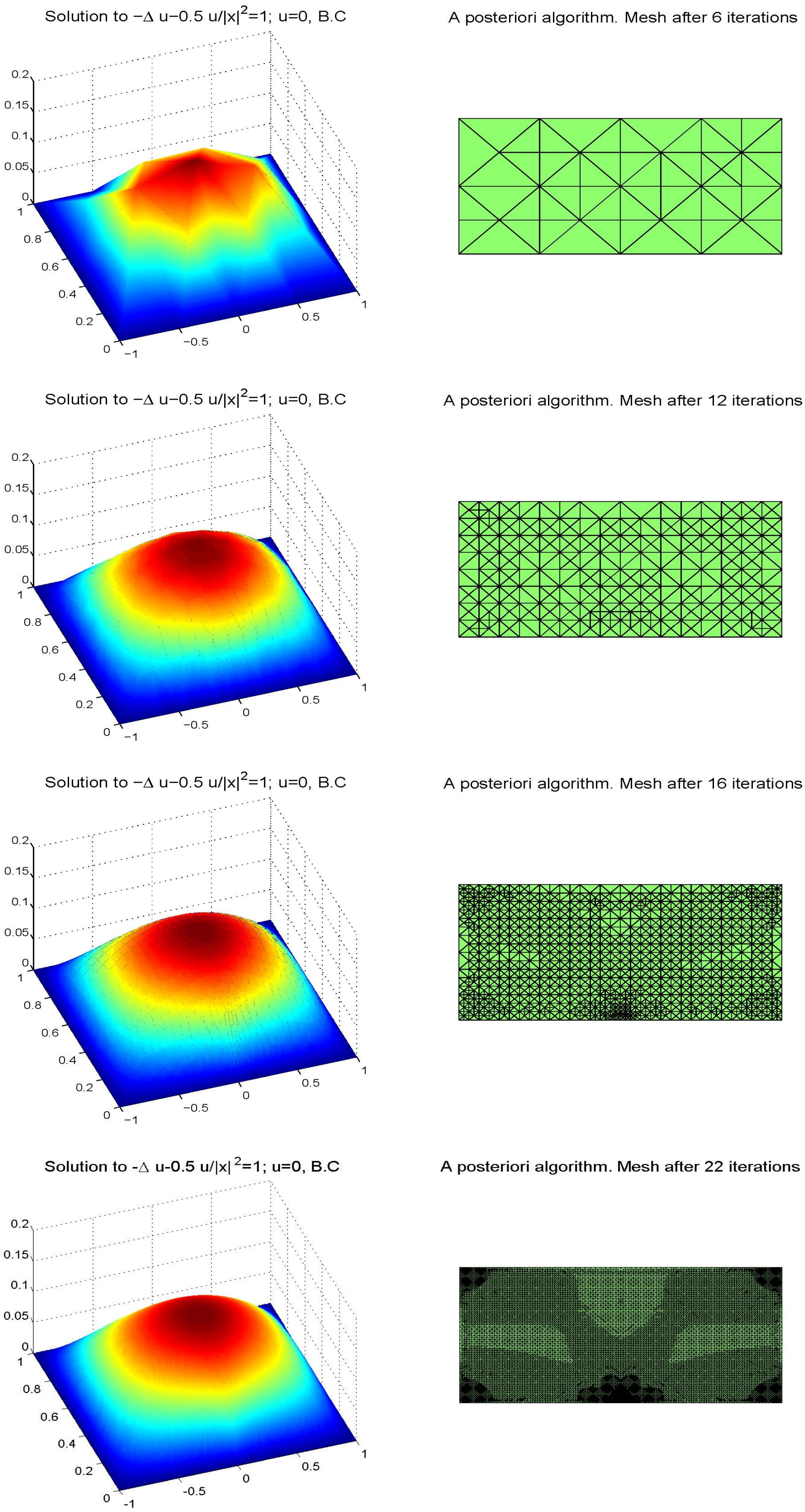

Figure 6.8: The case $\lambda = 0.5$ (Subcritical case)

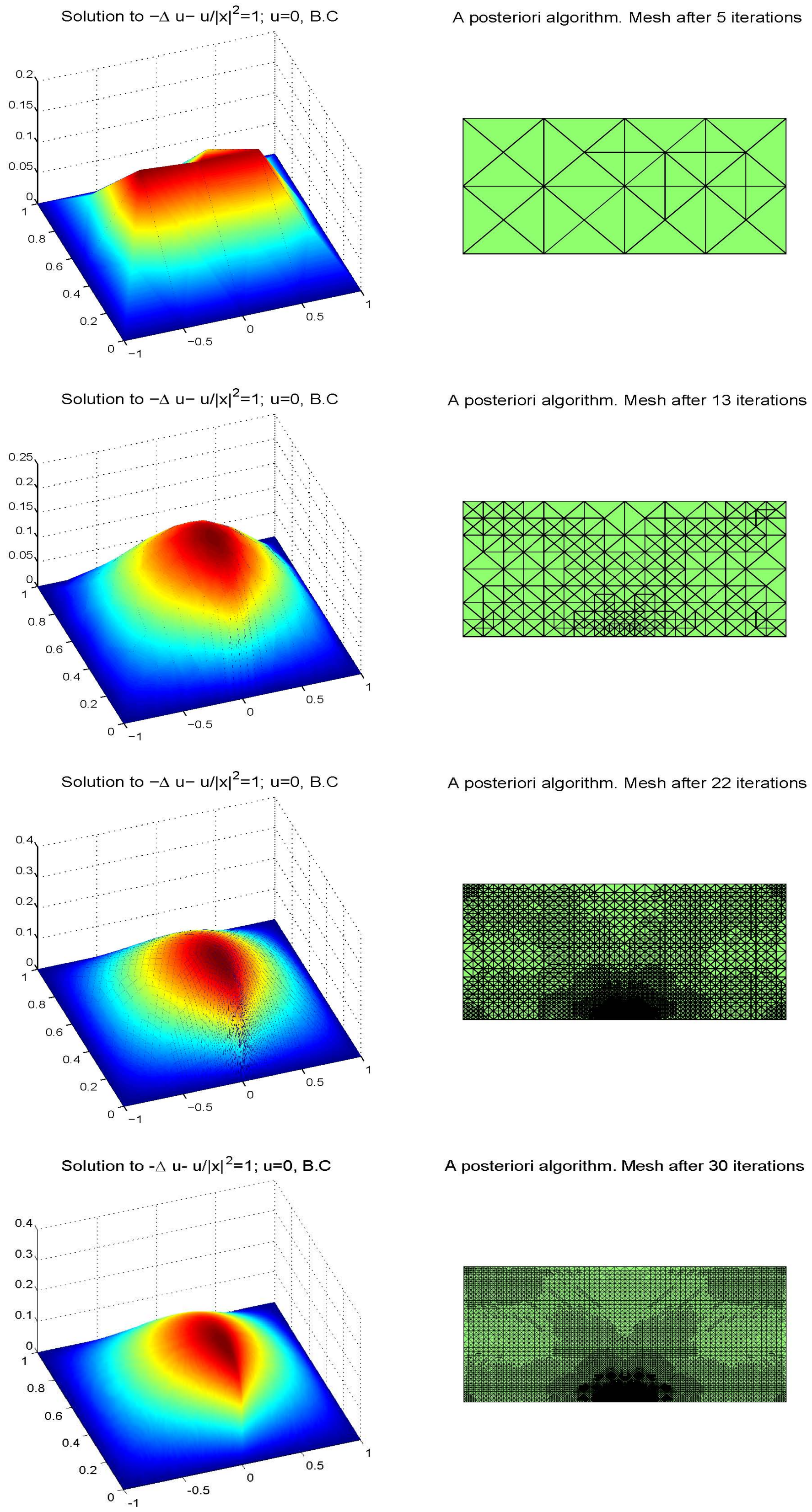

Figure 6.9: The case $\lambda = 1$ (Critical case)

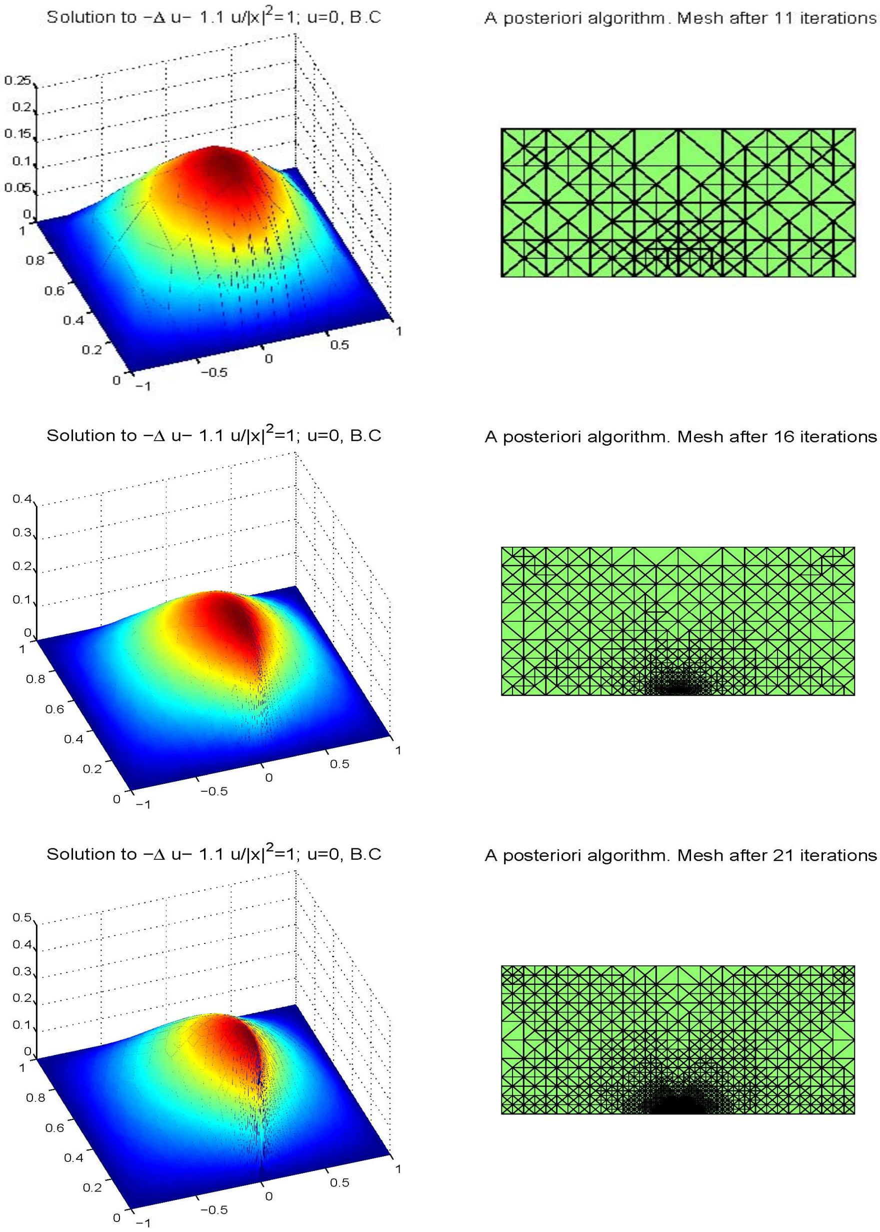

Figure 6.10: The case $\lambda = 1.1$ (Supercritical case-ill posedness)

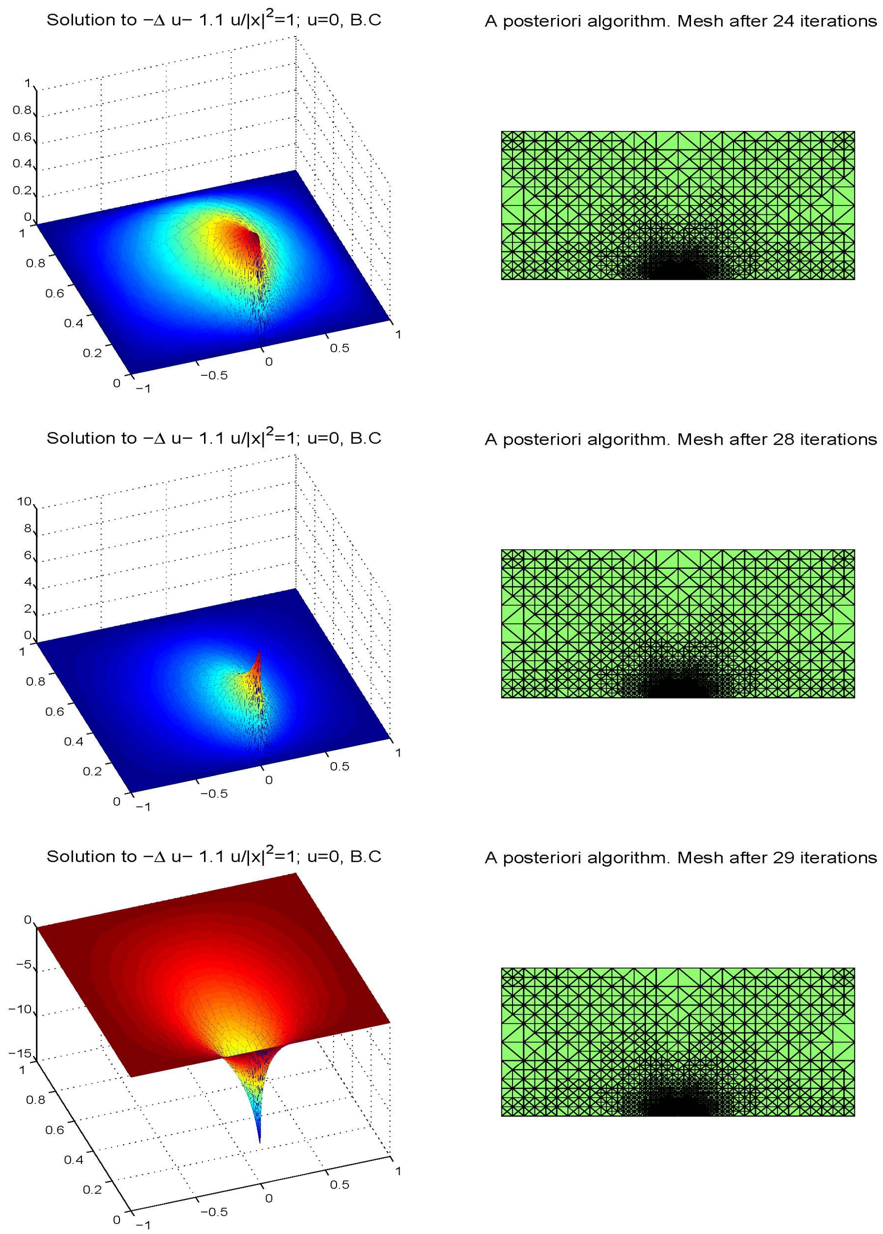

Figure 6.11: The case $\lambda = 1.1$ (Supercritical case-ill posedness)

Capítulo 7

Conclusiones y problemas abiertos

1. Desigualdades en dominios regulares y términos de resto óptimos

En la primera parte del Capítulo 2 hemos estudiado desigualdades de Hardy con un potencial singular localizado en la frontera de un dominio regular. Hemos demostrado que en dominios convexos Ω, la constante óptima es $\mu(\Omega) = N^2/4$ y, independientemente de la geometría, el mismo valor se mantiene de manera local cerca de la singularidad $x = 0$, es decir, $\mu(\Omega \cap B_{r_0}(0)) = N^2/4$, donde $r_0 > 0$ es una constante positiva dependiendo de la geometría de Ω. También, hemos demostrado que hay casos en los cuales $\mu(\Omega) < N^2/4$, tal y como se muestra en la Proposición 2.3.4. Sin embargo, para cualquier dominio regular Ω que contiene el origen en el borde se tiene que

$$\forall v \in H_0^1(\Omega), \quad C\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx, \tag{7.0.1}$$

para una constante $C > 0$. En términos generales, la constante de Hardy salta desde $(N-2)^2/4$ hasta $N^2/4$ al moverse desde el interior hacia la frontera. Así, la constante $N^2/4$ es óptima salvo algunos términos de orden inferior en L^2. Entonces, en aquellos casos en que $\mu(\Omega) = N^2/4$, hemos abordado la cuestión de encontrar límites inferiores de la diferencia positiva

$$\int_\Omega |\nabla v|^2 dx - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2} > 0. \tag{7.0.2}$$

Como hemos mencionado en el Teorema 2.3.1, las desigualdades que obtuvimos implican límites inferiores en (7.0.2) de orden $\int_\Omega v^2/(x_N - \gamma|x'|^2)dx$ que, modulo una constante universal, domina el término $\int_\Omega v^2/|x|dx$. La demostración del Teorema 2.3.1 está basada en una transformación funcional adaptada a la frontera cerca del origen $x = 0$. En la prueba del Teorema 2.3.1 la idea está escondida en el hecho de que la función $\phi = (x_N - \gamma|x'|^2)|x|^{-N/2}$ utilizada en la transformación, satisface la desigualdad puntual

$$-\Delta\phi > \frac{N^2}{4}\frac{\phi}{|x|^2} + C(\gamma)\frac{\phi}{x_N - \gamma|x'|^2} \tag{7.0.3}$$

para una constante $C(\gamma)$ que es no-negativa si $\gamma \geq 0$ y negativa de otra manera. La idea de construir supersoluciones como en (7.0.3) se enfatiza en la identidad de la Proposición 2.2.1, Sección 2.2. Comparando los Teoremas 2.3.1 y 2.3.2, podemos observar que los resultados mejoran cuando se utilizan descomposiciones en armónicos esféricos, que son herramientas utilizadas en la demostración del Teorema 2.3.2. Más precisamente, la desigualdad del Teorema 2.3.2 admite un término de resto óptimo del orden $\int v^2/(|x|^2 \log^2(1/|x|))dx$.

Sin embargo, el Teorema 2.3.2 podría ser también demostrado con la metodología planteada de la Sección 2.2, en la Proposición 2.2.1, con la elección

$$\phi = x_N|x|^{-N/2}\Big(\log\frac{1}{|x|}\Big)^{-1/2}.$$

Como un problema abierto se puede plantear la cuestión de encontrar el valor de $\mu(\Omega)$ para aquellos dominios regulares que satisfacen $\mu(\Omega) < N^2/4$.

2. Desigualdades en dominios cónicos

En la Sección 2.3.2 hablamos de desigualdades óptimas en dominios cónicos $\mathcal{C}_\gamma \subset \mathbb{R}^N$, $N \geq 2, \gamma \in (0, \pi)$. Recordamos que la constante óptima se caracteriza por $\mu(\mathcal{C}_\gamma) = (N-2)^2/4+ \lambda_1(\gamma)$, donde $\lambda_1(\gamma)$ es el primer autovalor del operador de Laplace-Beltrami en la superficie $S^{N-1} \cap \mathcal{C}_\gamma$. En este caso particular, $\lambda_1(\gamma)$ se caracteriza por el problema de minimización

$$\lambda_1(\gamma) = \inf_{u\in\mathcal{A}} \frac{\int_0^\gamma u_x^2 \sin^{N-2} x dx}{\int_0^\gamma u^2 \sin^{N-2} x dx} \tag{7.0.4}$$

para algún conjunto admisible de distribuciones u. Como hemos mencionado en la introducción de la Tesis, el valor $\lambda_1(\gamma)$ es conocido para $N = 2$ y $N = 4$.

En el caso multidimensional $N \geq 3$, hemos demostrado cotas inferiores de $\lambda_1(\gamma)$ que en algunos de los casos mejoran los resultados en [30], [15]. Sin embargo, por nuestro conocimiento, el valor exacto de $\lambda_1(\gamma)$ es aún un problema abierto para cualquier dimensión distinta de $N = 2$ y $N = 4$.

3. Desigualdades multipolares

- Falta por analizar la optimalidad del resultado de la Proposición 2.4.1.

- La identidad (7.0.8) podría ser aplicada también para varias elecciones de ϕ corespondientes al Teorema 2.4.1. En particular, si eligimos $\phi = E^{1/2} = \prod_{i=1}^n E_i^{1/2}$ se puede deducir la siguiente desigualdad

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx - \frac{(N-2)^2}{2} \sum_{1 \leq i < j \leq n} \int \frac{(x-x_i)(x-x_j)}{|x-x_i|^2 |x-x_j|^2} u^2 dx, \tag{7.0.5}$$

para cualquier $u \in C_0^\infty(\mathbb{R}^N)$. A continuación, podemos preguntarnos si la desigualdad (2.4.2) se puede probar a través de (7.0.5) ya que el potencial de (7.0.5) tiene una singularidad cuadrática en cualquier polo $x_i \in \mathbb{R}^N$. En primer lugar, observamos que esto es posible para el caso subcrítico $\mu < (N-2)^2/4$, que permite obtener términos de resto en la norma L^2. Sin embargo, (7.0.5) no proporciona terminos de resto en L^2 mejores que en (2.4.2).

En el caso crítico $\mu = (N-2)^2/4$, no podemos obtener terminos de resto en L^2 a partir de (7.0.5) salvo que el término de orden más bajo en la desigualdad (7.0.5) tenga signo positivo en una pequeña vencindad de los polos singulares x_i. Más precisamente, la pregunta a responder es si para cualquier configuración de los polos de $x_1, \ldots, x_n$, existe $\varepsilon > 0$ suficientemente pequeño tal que

$$- \sum_{1 \leq i < j \leq n} \frac{(x-x_i)(x-x_j)}{|x-x_i|^2 |x-x_j|^2} \geq 0, \quad \forall x \in \cup_{i=1}^{n} B_\varepsilon(x_i)? \tag{7.0.6}$$

Desafortunadamente, (7.0.6) no es verdad. Para esto, a continuación damos un contraejemplo (Figura 7.1).

Vamos a considerar una configuración de tres polos de x_1, x_2, x_3 que determinan un triángulo equilátero con los vértices en x_i, $i \in \{1,2,3\}$ tales que

$$|x_i - x_j| = d > 0, \quad \forall i \neq j, \quad \forall i,j \in \{1,2,3\}.$$

Dado $\varepsilon > 0$, además consideramos $x_\varepsilon \in \mathbb{R}^3$ situado en la recta determinada por x_1 y x_3 de tal manera que $|x_\varepsilon - x_1| = \varepsilon$, $|x_\varepsilon - x_3| = \varepsilon + d$ (como en Fig. 7.1). Entonces ocurre

$$|x_\varepsilon - x_2|^2 = \varepsilon^2 + d^2 + \varepsilon d, \quad |x_\varepsilon - x_3| = (\varepsilon + d)^2.$$

En vista de esto, se puede obtener fácilmente que

$$- \sum_{1 \leq i < j \leq 3} \frac{(x_\varepsilon - x_i)(x_\varepsilon - x_j)}{|x_\varepsilon - x_i|^2 |x_\varepsilon - x_j|^2} < 0,$$

para $\varepsilon > 0$ suficientemente pequeño, lo cual contradice (7.0.6).

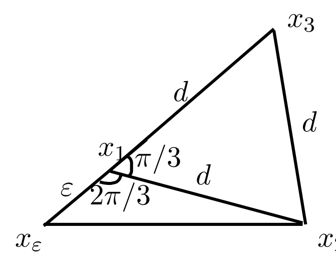

Figura 7.1: Contraejemplo para (7.0.6)

De manera más general, se puede demostrar que no hay ninguna configuración de polos $x_1, \ldots, x_n$ para cuales (7.0.6) sea verdad. La condición (7.0.6) no se cumple en los polos singulares x_{k_i}, con $\{k_i \mid i \in \{1, \ldots, n\}\} \subset \{1, \ldots, n\}$, que se encuentran en la frontera de la clausura convexa de los polos $x_1, \ldots, x_n$.

- A pesar de que hemos demostrado desigualdades óptimas para los potenciales multipolares con singularidades críticas como en el Teorema 2.4.1, uno de los problemas más difíciles que todavía está por demostrar, es el que se refiere a la optimización de la constante λ en

$$\int_\Omega |\nabla u|^2 dx \geq \lambda \sum_{i=1}^{n} \int_\Omega \frac{u^2}{|x - x_i|^2} dx,$$

para cualquier configuración de los polos $x_1, x_2, \ldots, x_n \in \mathbb{R}^N$, con $N \geq 3$, $n \geq 2$.

4. Número infinito de singularidades

- Con el mismo espíritu que en (7.0.8) se trata de buscar potenciales V, con un número infinito de singularidades para que la desigualdad de Hardy sea cierta. Además, una vez que esos V vienen dados por una serie infinita, primero nos tenemos que asegurar de que estan bien definidos. Por ejemplo, un potencial de la forma

$$V(x) = \sum_{(i,j,k) \in \mathbb{Z}^3} \frac{1}{|x_1 - i|^2 + |x_2 - j|^2 + |x_3 - k|^2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

  diverge en cada punto x $\in \mathbb{R}^3$ y por tanto, la desigualdad de Hardy correspondiente no tiene sentido.

Hasta ahora, hemos obtenido resultados positivos para desigualdades, que implican potenciales con un número infinito de singularidades distribuidas periódicamente sobre una recta en $\mathbb{R}^N$, $N \geq 3$, en dominios cilíndricos.

A continuación se discute la posibilidad de obtener constantes óptimas para nuestras desigualdades.

Para ser más precisos, consideramos el caso de una recta N-dimensional definida por

$$l_N := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_2 = x_3 = \dots = x_N = 0\}.$$

y el cilindro asociado

$$\mathcal{C}_{1,l_N} := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_2^2 + \dots x_N^2 < 1\},$$

en la que las singularidades $\{p_i\}_{i\in\mathbb{Z}}$ están dadas por

$$p_i = ie_1, \quad \forall i \in \mathbb{Z}, \quad e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N.$$

Hemos demostrado que existe una constante positiva $C = C(N) > 0$ tal que

$$\int_{\mathcal{C}_{1,l_N}} |\nabla u|^2 \geq C \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{1,l_N}} V(x) u^2 dx, , \quad \forall u \in H_0^1(\mathcal{C}_{1,l_N}), \tag{7.0.7}$$

donde la expresión de V es

$$V(x) := \sum_{i\in\mathbb{Z}} \frac{1}{|x - p_i|^2} = \sum_{i\in\mathbb{Z}} \frac{1}{|x_1 - i|^2 + x_2^2 + \dots + x_N^2}$$

Esto se hizo mediante la desigualdad más general

$$\int_{\mathcal{C}_{1,l_N}} |\nabla u|^2 dx \geq \int_{\mathcal{C}_{1,l_N}} \Big(- \frac{\Delta\phi}{\phi} \Big) u^2 dx, \quad \forall u \in H_0^1(\mathcal{C}_{1,l_N}), \tag{7.0.8}$$

que es válida para cualquier función $\phi \in C^2(\mathcal{C}_{1,l_N} \setminus d_N)$.

Por lo tanto, nuestro problema se reduce a construir (ϕ, λ) de tal manera que

$$-\frac{\Delta\phi}{\phi} \geq \lambda V(x) \quad \text{a.e. in } \mathcal{C}_{1,l_N}.$$

Una pregunta interesante que podemos plantear es la de construir pares (ϕ, λ) que cumplan la identidad

$$- \frac{\Delta\phi}{\phi} = \lambda V(x) \quad \text{a.e. in } \mathcal{C}_{1,l_N}? \tag{7.0.9}$$

Algunos consejos y dificultades

En primer lugar, observamos que el potencial $V(x)$ se puede escribir como

$$V(x) = \sum_{i\in\mathbb{Z}} \frac{1}{(i - a)^2 + \rho^2}, \tag{7.0.10}$$

donde $a = x_1$ y $\rho = \sqrt{x_2^2 + \ldots x_N^2}$.

Usando elementos de análisis complejo, se puede demostrar que

$$V(x) = \frac{\pi}{\rho} \frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}. \tag{7.0.11}$$

Dado que estamos trabajando en los cilindros, escribimos el Laplaciano en coordenadas cilíndricas, coordenadas, es decir,

$$\Delta = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\theta\theta} + \partial_{x_1x_1}. \tag{7.0.12}$$

Entonces, debido a (7.0.11) y (7.0.12), nuestra ecuación (7.0.9) se convierte en

$$\partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho\phi + \frac{1}{\rho^2}\partial_{\theta\theta}\phi + \partial_{x_1x_1}\phi + \lambda\frac{\pi}{\rho}\frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}\phi = 0, \tag{7.0.13}$$

donde $\phi(x) = \phi(\rho, x_1, \theta)$, con $\rho \in (0,1)$, $x_1 \in \mathbb{R}$, $\theta \in (0, 2\pi)$. Además, el sistema (7.0.13) admite las condiciones iniciales

$$\phi_{|\rho=1} = 0, \quad \partial_\rho\phi_{|\rho=0} = 0.$$

Como el potencial V no depende de θ, vamos a buscar ϕ independiente de θ. En este caso, (7.0.13) llega a ser

$$\partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho\phi + \partial_{x_1x_1}\phi + \lambda\frac{\pi}{\rho}\frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}\phi = 0. \tag{7.0.14}$$

La posibilidad de resolver la ecuación (7.0.14) podría ser una pista para encontrar la constante óptima en (7.0.7).

Capítulo 3

Las desigualdades de Hardy (7.0.1) son más débiles pero juegan un papel crucial en el estudio del controlabilidad de las ecuaciones de ondas con potenciales cuadráticos singulares. En ese contexto, uno puede despreciar de los términos de resto en L^2, utilizando argumentos de compacidad-unicidad (véase [40]).

Más precisamente, en el Capítulo 3 hemos enfatizado el papel de la identidad de Pohozaev con singularidad úbicada en la frontera, al estudiar la controlabilidad de los sistemas conservativos sistemas como ecuaciones de ondas y de Schrödinger. Hemos demostrado que para cualquier $\lambda \leq \lambda(N) = N^2/4$, los sistemas correspondientes son observables desde Γ_0 preciso en (3.1.13). Nuestro resultado aumenta el rango de valores $\lambda \leq (N-2)^2/4$, que fue demostrado en [109] en el contexto de singularidades interiores, para el cual se tiene el resultado de control.

1. Las restricciones geométricas

La presunción geométrica de Γ_0 es realmente necesaria, si no nuestra prueba no funciona. Aún está abierto el caso de considerar el dominio de control Γ_{x_0} cuyo centro de gravedad esta centrado en un punto x_0 diferente de cero, es decir, $\Gamma_{x_0} = \{x \in \Gamma \mid (x - x_0) \cdot \nu \geq 0\}$. Esta eleción de Γ_{x_0} dan lugar a algunas dificultades técnicas que también han sido enfatizadas en [109]. Para concluir, la prueba en el caso general Γ_{x_0} debe aplicar una técnica diferente de la que hemos utilizado hasta ahora.

2. Singularidades multipolares

La identidad de Pohozaev y el problema de control que hemos estudiado, podrían ser planteados para operadores más complicados, como por ejemplo $L = -\Delta - V(x)$, donde $V(x)$ denota un potencial multipolar. Dentro de los problemas más interesantes nos referimos al caso de potenciales bipolares. Sería interesante analizar el proceso límite cuando una partícula colapsa a la otra, y como esto aplica en el contexto de la controlabilidad y los procesos de difusión para la ecuación del calor, discutiendo el tiempo de decaída de las soluciones.

Capítulo 4

En el Teorema 4.1.1 probamos la controlabilidad nula para la ecuación del calor con un potencial cuadrático úbicado en la frontera. Para un dominio cóncavo, probamos que el sistema puede ser controlable a cero, actuando con un control distribuido en cualquier subconjunto abierto $\omega \subset \Omega$. Por el contrario, para un dominio convexo, en el Teorema 4.1.2 hemos visto que la controlabilidad se tiene siempre que la región del control rodea la singularidad. La pregunta sobre si la restricción anterior se puede quitar o no es un problema desconocido para nosotros.

Capítulo 5

En el Capítulo 5 hemos estudiado la regularidad de la ecuación $-\Delta u - \lambda/|x|^2 u = f$ con condiciones de frontera de Dirichlet y el dato $f \in L^2(\Omega)$.

En las situaciones en las que la solución es radial (incluido el caso de una sola dimensión), en los Teoremas 5.2.3, 5.3.3 hemos demostrado resultados óptimos de regularidad en los espacios de Sobolev fraccionarios $H^{1+s}(\Omega)$ donde $s = s(\lambda) < 1$ con λ suficientemente cerca de la constante crítica de Hardy. De lo contrario, si λ esta lejos de la constante crítica, la solución pertenece a $H^{2-\varepsilon}(\Omega)$ (para cualquier pequeño $\varepsilon > 0$) en una sola dimensión, respectivamente, a H^2 en el caso multidimensional.

Una de las cuestiones que quedan para estudiar en el caso de una sola dimensión, es la regularidad H^2 para valores de λ suficientemente negativos.

Además, aunque hemos podido comprobar el comportamiento asintótico de la parte radial de las soluciones no radiales, la regularidad de las soluciones no-radiales aún es un problema abierto. Sin embargo, dado que el operador que participa en la ecuación es radial, se espera que las soluciones no radiales hereden las propiedades de regularidad de las soluciones radiales.

Capítulo 6

En la Sección 6.3.4 hemos analizado el rendimiento de los métodos de elementos finitos para la ecuación $-\Delta u - \lambda/|x|^2 u = f$ con condiciones de frontera de Dirichlet y el dato $f \in L^2(\Omega)$.

En el caso de una sola dimensión, hemos construido mallas adaptadas, usando estimaciones a priori, y hemos obtenido tasas de convergencia óptimas para las aproximaciones de elementos finitos. En el caso bidimensional, hemos desarrollado algunos códigos numéricos basados en un análisis a posteriori para la solución numérica y el dato f. En consecuencia, hemos diseñado soluciones numéricas adaptadas que alcanzan un error tolerado utilizando un número mínimo de iteraciones. A pesar de ellos, un análisis completo de las aproximaciones de elementos finitos para problemas con singularidades aún esta por ser completado. las metas y objetivos más factibles son:

1. completar rigurosamente los resultados teóricos sobre la convergencia y la complejidad del algoritmo de AFEM para los operadores con potenciales de cuadrados inversos.

2. analizar el caso crítico que corresponde a la crítica constante de Hardy λ, en la que el espacio de la energía se escapa del clásico espacio de Sobolev H^1.

3. comparar la compatibilidad de los resultados numéricos y teóricos.

4. extender el análisis adaptativo a los problemas de evolución (por ejemplo, las ecuaciónes de ondas y calor).

Chapter 7

Conclusions and open problems

Chapter 2

1. Inequalities in smooth-domains and sharp reminder terms

In the first part of Chapter 2 we studied Hardy inequalities with one singular potential located on the boundary of a smooth domain. We proved that in convex domains Ω the optimal constant is $\mu(\Omega) = N^2/4$ and, independently on the geometry, the same value is maintained close to the singularity $x = 0$, i.e. $\mu(\Omega \cap B_{r_0}(0)) = N^2/4$, for some positive constant $r_0 > 0$, depending on the geometry of Ω. We also proved that there are cases in which $\mu(\Omega) < N^2/4$ as shown in Proposition 2.3.4. However, for any smooth domain Ω containing the origin on the boundary it holds

$$\forall v \in H_0^1(\Omega), \quad C\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx \geq \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2}dx, \tag{7.0.1}$$

for some constant $C > 0$. Roughly speaking, the Hardy constant jumps from $(N-2)^2/4$ to $N^2/4$ when moving the singularity from interior on the boundary. Thus, the constant $N^2/4$ is optimal in the Hardy inequality, up to some lower order terms in L^2. Then, in those cases where $\mu(\Omega) = N^2/4$, we addressed the question of finding lower bounds for the positive difference

$$\int_\Omega |\nabla v|^2 dx - \frac{N^2}{4}\int_\Omega \frac{v^2}{|x|^2} > 0. \tag{7.0.2}$$

As we mentioned in Theorem 2.3.1, the inequalities we obtained involve lower bounds in (7.0.2) of order $\int v^2/(x_N - \gamma|x|^2)dx$ which, up to some universal constant, dominate the term $\int v^2/|x|dx$. The proof of Theorem 2.3.1 is based on a functional transformation adapted to the boundary near the origin $x = 0$. In the proof of Theorem 2.3.1 the idea is hidden in the fact that the function $\phi = (x_N - \gamma|x'|^2)|x|^{-N/2}$ used in the transformation, satisfies the pointwise inequality

$$-\Delta\phi > \frac{N^2}{4}\frac{\phi}{|x|^2} + C(\gamma)\frac{\phi}{x_N - \gamma|x'|^2} \tag{7.0.3}$$

for some positive $C(\gamma)$ which is non-negative if $\gamma \geq 0$ and negative otherwise. The idea of building supersolutions as in (7.0.3) was emphasized in the identity of Proposition 2.2.1 in Section 2.2.

Comparing Theorems 2.3.1-2.3.2, we can remark that the results improve when using spherical harmonics decomposition which are tools used in the proof of Theorem 2.3.2. More precisely, the inequality stated in Theorem 2.3.2 admits an optimal reminder term of order $\int v^2/(|x|^2 \log^2(1/|x|))dx$. Thus, spherical harmonics decomposition yields better results.

However, Theorem 2.3.2 could be also proved in the spirit of Section 2.2, Proposition 2.2.1, with the choice

$$\phi = x_N|x|^{-N/2}\Big(\log\frac{1}{|x|}\Big)^{-1/2}.$$

As an open problem we can address the question to finding the value $\mu(\Omega)$ for those smooth domains satisfying $\mu(\Omega) < N^2/4$.

2. Inequalities in conical domains

In Section 2.3.2 we discussed optimal inequalities in conical domains $\mathcal{C}_\gamma \subset \mathbb{R}^N$, $N \geq 2$, $\gamma \in (0,\pi)$. We recall that the optimal constant is characterized by $\mu(\mathcal{C}_\gamma) = (N-2)^2/4 + \lambda_1(\gamma)$, where $\lambda_1(\gamma)$ is the first eigenvalue of the Laplace-Beltrami operator on the surface $S^{N-1}\cap\mathcal{C}_\gamma$. In this particular case, $\lambda_1(\gamma)$ is characterized by the minimization problem

$$\lambda_1(\gamma) = \inf_{u\in\mathcal{A}} \frac{\int_0^\gamma u_x^2 \sin^{N-2} x dx}{\int_0^\gamma u^2 \sin^{N-2} x dx} \tag{7.0.4}$$

for some admissible set of distributions $\mathcal{A}$. As we mentioned at the beginning of the Tesis, $\lambda_1(\gamma)$ is known for $N=2$ and $N=4$.

In the multi-dimensional case, $N \geq 3$, we proved several qualitative lower bounds for $\lambda_1(\gamma)$, which in some of the cases improve the results in [30], [15]. However, to the best of our knowledge, the optimality is still an open problem for any dimension excepting $N=2$ and $N=4$.

3. Multipolar inequalities

- The optimality of the result in Proposition 2.4.1 remains to be analyzed.
- Identity (7.0.8) could be also applied for other choices of ϕ that we performed in Theorem 2.4.1. In particular, If we choose $\phi = E^{1/2} = \prod_{i=1}^n E_i^{1/2}$ then we deduce the following inequality

$$\int |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \sum_{i=1}^{n} \int \frac{u^2}{|x-x_i|^2} dx - \frac{(N-2)^2}{2} \sum_{1 \leq i < j \leq n} \int \frac{(x-x_i)(x-x_j)}{|x-x_i|^2 |x-x_j|^2} u^2 dx, \tag{7.0.5}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Next we may wonder the question if we can reprove inequality (2.4.2) through (7.0.5) since the potential in (7.0.5) has a critical quadratic singularity at any pole $x_i \in \mathbb{R}^N$. Firstly, we note that this is possible for the subcritical case $\mu < (N-2)^2/4$ which allows to get lower order terms in L^2-norm. However, (7.0.5) does not provide better L^2 lower order term than (2.4.2) does.

In the critical case $\mu = (N-2)^2/4$ we cannot obtain L^2 reminder terms from (7.0.5) unless the lower order term in inequality (7.0.5) has positive sign in a small neighborhood of the singular poles x_i. More precisely, the question to answer is whether for any configuration of the poles $x_1, \ldots, x_n$, there exists $\varepsilon > 0$ small enough such that

$$- \sum_{1 \leq i < j \leq n} \frac{(x-x_i)(x-x_j)}{|x-x_i|^2 |x-x_j|^2} \geq 0, \quad \forall x \in \cup_{i=1}^n B_\varepsilon(x_i)? \tag{7.0.6}$$

Unfortunately, (7.0.6) is not true. We give a counterexample below (Fig. 7.1).

Let us consider a configuration of three poles x_1, x_2, x_3 determining an equilateral triangle with the vertices at x_i, $i \in \{1,2,3\}$ such that

$$|x_i - x_j| = d > 0, \quad \forall i \neq j, \quad \forall i, j \in \{1,2,3\}.$$

Given $\varepsilon > 0$, we also consider $x_\varepsilon \in \mathbb{R}^3$ located on the line determined by x_1 and x_3 such that $|x_\varepsilon - x_1| = \varepsilon$, $|x_\varepsilon - x_3| = \varepsilon + d$ (as in Fig. 7.1). Then we have

$$|x_\varepsilon - x_2|^2 = \varepsilon^2 + d^2 + \varepsilon d, \quad |x_\varepsilon - x_3| = (\varepsilon + d)^2.$$

In view of this, we can easily obtain that

$$- \sum_{1 \leq i < j \leq 3} \frac{(x_\varepsilon - x_i)(x_\varepsilon - x_j)}{|x_\varepsilon - x_i|^2 |x_\varepsilon - x_j|^2} < 0,$$

for $\varepsilon > 0$ small enough, fact which contradicts (7.0.6).

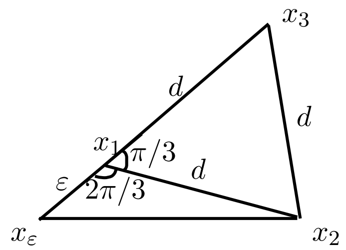

Figure 7.1: Counterexample to (7.0.6)

More general, one can show that there is no configuration $x_1, \dots, x_n$ for which (7.0.6) is true. The condition (7.0.6) is violated at the singular poles x_{k_i}, with $\{k_i \mid i \in \{1, \dots, n\}\} \subset \{1, \dots, n\}$, which are located on the boundary of the smallest convex set containing all the poles $x_1, \dots, x_n$.

- Although we proved optimal inequalities for multipolar potentials with quadratic singularities as in Theorem 2.4.1, one of the most challenging problems which is still to be proved, refers to the optimality of the constant λ in

$$\int_\Omega |\nabla u|^2 dx \geq \lambda \sum_{i=1}^n \int_\Omega \frac{u^2}{|x - x_i|^2} dx,$$

for each configuration for the poles $x_1, x_2, \dots, x_n \in \mathbb{R}^N$, with $N \geq 3$, $n \geq 2$.

4. Infinite number of singularities

- In the spirit of (7.0.8) we may look for potentials V with an infinite number of singularities for which the Hardy inequality holds true. Besides, as such V are given as an infinite series one needs to make sure that they are well-defined. For instance, a potential of the form

$$V(x) = \sum_{(i,j,k) \in \mathbb{Z}^3} \frac{1}{|x_1 - i|^2 + |x_2 - j|^2 + |x_3 - k|^2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

  diverges at every point $x \in \mathbb{R}^3$ and therefore the corresponding Hardy inequality does not make sense.

So far, we obtained positive results for Hardy inequalities involving potentials with infinite number of singularities periodically distributed on a line in $\mathbb{R}^N$, $N \geq 3$, in cylindrical domains.

Next we discuss the possibility to obtain optimal constants in our inequalities.

To be more precise, here we consider the case of the N-d line

$$l_N := \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_2 = x_3 = \ldots = x_N = 0\}.$$

and the associated cylinder

$$\mathcal{C}_{1,l_N} := \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_2^2 + \ldots x_N^2 < 1\},$$

in which the singularities $\{p_i\}_{i\in\mathbb{Z}}$ are given by

$$p_i = ie_1, \quad \forall i \in \mathbb{Z}, \quad e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N.$$

We have proved that there exists a positive constant $C = C(N) > 0$ such that

$$\int_{\mathcal{C}_{1,l_N}} |\nabla u|^2 \geq C \sum_{i\in\mathbb{Z}} \int_{\mathcal{C}_{1,l_N}} V(x) u^2 dx, , \quad \forall u \in H_0^1(\mathcal{C}_{1,l_N}), \tag{7.0.7}$$

where V is given by

$$V(x) := \sum_{i\in\mathbb{Z}} \frac{1}{|x - p_i|^2} = \sum_{i\in\mathbb{Z}} \frac{1}{|x_1 - i|^2 + x_2^2 + \ldots + x_N^2}$$

This was done by using the more general inequality

$$\int_{\mathcal{C}_{1,l_N}} |\nabla u|^2 dx \geq \int_{\mathcal{C}_{1,l_N}} \Big(- \frac{\Delta\phi}{\phi} \Big) u^2 dx, \quad \forall u \in H_0^1(\mathcal{C}_{1,l_N}), \tag{7.0.8}$$

which holds for any function $\phi \in C^2(\mathcal{C}_{1,l_N} \setminus d_N)$.

So, our problem reduced to build (ϕ, λ) such that

$$-\frac{\Delta\phi}{\phi} \geq \lambda V(x) \quad \text{ a.e. in } \mathcal{C}_{1,l_N}.$$

We did this for some range of constants λ and then we considered the largest admissible λ which we called $C = \sup \lambda$ in (7.0.7).

An interesting question we can address is wether we can build pairs (ϕ, λ) to satisfy the identity

$$- \frac{\Delta\phi}{\phi} = \lambda V(x) \quad \text{ a.e. in } \mathcal{C}_{1,l_N}? \tag{7.0.9}$$

Some hints and difficulties

Firstly, we note that the potential $V(x)$ can be written as

$$V(x) = \sum_{i\in\mathbb{Z}} \frac{1}{(i - a)^2 + \rho^2}, \tag{7.0.10}$$

where $a = x_1$ and $\rho = \sqrt{x_2^2 + \dots x_N^2}$.

Using complex analysis we can show that

$$V(x) = \frac{\pi}{\rho} \frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}. \tag{7.0.11}$$

Since we are working in cylinders we write the Laplacian in cylindrical coordinates, that is,

$$\Delta = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\theta\theta} + \partial_{x_1x_1}. \tag{7.0.12}$$

Then, due to (7.0.11) and (7.0.12) our equation (7.0.9) becomes

$$\partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho\phi + \frac{1}{\rho^2}\partial_{\theta\theta}\phi + \partial_{x_1x_1}\phi + \lambda\frac{\pi}{\rho}\frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}\phi = 0, \tag{7.0.13}$$

where $\phi(x) = \phi(\rho, x_1, \theta)$, with $\rho \in (0,1)$, $x_1 \in \mathbb{R}$, $\theta \in (0, 2\pi)$. The system (7.0.13) admits the initial conditions

$$\phi_{|\rho=1} = 0, \quad \partial_\rho\phi_{|\rho=0} = 0.$$

Since the potential V does not depend on θ we will look for ϕ independent of θ. In this case, (7.0.13) becomes

$$\partial_{\rho\rho}\phi + \frac{1}{\rho}\partial_\rho\phi + \partial_{x_1x_1}\phi + \lambda\frac{\pi}{\rho}\frac{e^{2\pi\rho} - e^{-2\pi\rho}}{e^{2\pi\rho} + e^{-2\pi\rho} - 2\cos(2\pi x_1)}\phi = 0, \tag{7.0.14}$$

The possibility to solve the equation (7.0.14) could help to find the optimal constant in (7.0.7).

Chapter 3

Weaker Hardy inequalities (7.0.1) play a crucial role when studying the controllability of wave equations with quadratic singular potentials. In that setting, one can get rid of the L^2-reminder terms, using compactness-uniqueness arguments (see [40]).

More precisely, in Chapter 3 we have shown the role of the Pohozaev identity, in the context of boundary singularities, when studying the controllability of conservative systems like wave and Schrödinger equations. We proved that for any $\lambda \leq \lambda(N) = N^2/4$, the corresponding systems are exact observable from Γ_0 precised in (3.1.13). Our result enlarges the range of values $\lambda \leq (N-2)^2/4$ for which the control holds, proved firstly in [109] in the context of interior singularities.

1. Geometric constraints

The geometrical assumption for Γ_0 is really necessary, otherwise our proof does not work. Of course, it is still open to be analysed the case when the central of gravity of Γ_{x_0} is centered at a point x_0 different by zero, i.e. $\Gamma_{x_0} = \{x \in \Gamma \mid (x - x_0) \cdot \nu \geq 0\}$. This choice of Γ_{x_0} provide some technical difficulties which have been also emphasized in [109]. En eventually proof in the case of a such domain Γ_{x_0} should apply a different technique that we have used so far.

2. Multipolar singularities

The same Pohozaev identity and controllability issues could be address for more complicated operators, like for instance $L = -\Delta - V(x)$, where $V(x)$ denotes a multi-particle potential. To the best of our knowledge, even if there are some important works studying Hardy-type inequalities for multipolar potentials an accurate analysis is still to be done. On the of the most interesting problems refers to the case of two particles systems. It would be interesting to analyze the limit process when one particle collapses to the other, and to apply this in the context of controllability and the diffusion heat processes discussing the time decay of solutions.

Chapter 4

In Theorem 4.1.1 we proved the null-controllability of the heat equation with boundary quadratic singular potential. For a concave domain we showed that the system can be leaded to the zero state acting with a control distributed in any open subset $\omega \subset \Omega$. On the contrary, for a convex domain we emphasized in Theorem 4.1.2 that the null-controllability holds provided the control region surrounds the singularity. The question wether the above constraint on the control region can removed is still unknown for us.

Chapter 5

In Chapter 5 we proved regularity results for the equation $-\Delta u - \lambda/|x|^2 u = f$ with Dirichlet boundary conditions and data $f \in L^2(\Omega)$.

In the situations when the solution is radial (including the one dimensional case), in

Theorems 5.2.3, 5.3.3 we proved optimal regularity results in the fractional Sobolev spaces $H^{1+s}(\Omega)$ for some constant $s = s(\lambda) < 1$, for λ sufficiently close to the critical Hardy constant. Otherwise, if λ is very negative, the solution belongs to $H^{2-\varepsilon}(\Omega)$ (for any small $\varepsilon > 0$) in the one dimensional respectively to H^2 in the multidimensional case.

One further question concerns to study if the solution is allowed to have H^2-regularity in the one dimensional case.

In addition, although we proved the asymptotic behavior of the radial part of the non-radial solutions, the regularity of non-radial solutions is still an open problem. However, since the operator involved in the equation is radial, we expect that the non-radial solutions inherit the regularity properties of the radial solutions.

Chapter 6

In Section 6.3.4 we analyzed the performance of the finite element schemes to the equation $-\Delta u - \lambda/|x|^2 u = f$ with Dirichlet boundary conditions.

In the one dimensional case, we built adapted mesh refinements, based on a priori estimates, to obtain optimal convergence of the finite element approximations.

In the two-dimensional case, we developed some numerical codes based on an posteriori analysis for the numerical solution and the data. In consequence, we designed adapted numerical solutions which reach to a prescribed error using a minimum number of iterations.

Despite of these, a full analysis of the finite element approximations to problems with singularities is still to be completed. In the following we state further possible goals and objectives:

1. complete rigourously the theoretical results about the convergence and complexity of the algorithm of AFEM for operators with inverse square potentials

2. Analyze the critical case corresponding to the critical Hardy constant λ, in which the error energy space escapes from the classical Sobolev space H^1

3. compare the compatibility of the numerical and theoretical results

4. extend this analysis to evolution problems (e.g. wave and heat like equations)

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